Math 2602 **Exam 1** Fall 2014

Instructor: Sal Barone

Name: KEY

GT username: ______________________

Circle your TA/section: (G1) Conrad (G2) Andy (G3) Marcel (G4) Dylan

1. No books or notes are allowed.
2. You may use ONLY NON-GRAPHING and NON-PROGRAMABLE scientific calculators. All other electronic devices are not allowed.
3. Show all work and fully justify your answer to receive full credit.
4. Good luck!

Page	Max. Possible	Points
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3	24	
4	25	
Total	100	

1. Over a total of 11 years of teaching I have had 1,500 students. Find the largest integer n such that at least n of my previous students have the same initials. Recall, a person's initials are a string of the form $X.Y.$ where X is the first letter of their first name and Y is the first letter of their last name (for example, my initials are S.B.). (12 pts.)

There are $26 * 26 = 676$ possible initials, and $\left\lceil \frac{1500}{676} \right\rceil = \lceil 2.2189 \rceil = 3$. So, by the strong pigeonhole principle there are at least 3 Students with the same initials. ∎

2. Prove for all integers $n \geq 1$ that (15 pts.)

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}.$$

By induction.

Base case $n=1$: $\frac{1}{2} \stackrel{?}{=} \frac{2^1 - 1}{2^1} = \frac{2-1}{2} = \frac{1}{2}$ ✓

Induction step: Suppose $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}$.

Then $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}}$

$$= \frac{2(2^k - 1) + 1}{2^{k+1}} = \frac{2^{k+1} - 2 + 1}{2^{k+1}}$$

$$= \frac{2^{k+1} - 1}{2^{k+1}}.$$ ✓

This completes the proof. ∎

3. Determine if the argument is valid or invalid. Prove your answer. (12 pts. each)

(a)

$$\begin{array}{c} q \longrightarrow [\neg(p \to r)] \\ \neg(\neg r \wedge s) \\ s \wedge p \\ \hline \neg q \end{array}$$

Valid. Suppose not. Then there is an assignment such that the conclusion is false and all the assumptions are true. In such an assignment $\neg q$ is false, hence q is true. By the first assumption $\neg(p \to r)$ is true. So p is true and r is false. Since r is false and $\neg(\neg r \wedge s)$ is true, s must be false. But if s is false then $s \wedge p$ is false. This contradicts the fact that the assumptions are all true in the assignment we are considering. So, the argument is valid.

(b)

$$\begin{array}{c} p \longrightarrow (q \wedge r) \\ r \longrightarrow p \\ \hline \neg q \end{array}$$

Invalid. When p is true, q is true, and r is true we have

$p \to (q \wedge r)$ is true $T \to (T \wedge T)$ ✓

$r \longrightarrow p$ is true $T \to T$ ✓

$\neg q$ is false ✓

4. Short answer section: put a number in each box. You may use the factorial function $n!$ and binomial coefficient $\binom{n}{k}$ without simplifying. Please show your work for potential partial credit. (8 pts. each)

(i) The number of 5-letter words (not necessarily real English words) containing the string R-A-P, where the letters are consecutive and in order, is

$\boxed{3978}$

$x \neq y$ $\quad x(RAP)y$ $\quad 3! = 6$ permutations, 26 choices for x, 25 choices for y $\quad > 3900$

$x = y$ $\quad x(RAP)x$ $\quad 3$ permutations, 26 choices for x $\quad > 78$

(ii) A license plate is a string of 7 characters. How many license plates can be made if the license plate needs to contain 3 different letters and 4 different numbers? (e.g., B5A91Z4)

$\boxed{\binom{26}{3}\binom{10}{4}\cdot 7! = 2,751,840,000}$

$$\underbrace{\binom{26}{3}}_{\text{pick 3 letters}} * \underbrace{\binom{10}{4}}_{\text{pick 4 numbers}} * \underbrace{7!}_{\text{7! permutations of 7 unique characters}} = 2600 * 210 * 5040$$

(iii) How many license plates can be made if the license plate needs to contain 3 different letters and 4 different numbers and the letters must all be to the left of all the numbers, and furthermore the numbers must be increasing from left to right and the letters must be in alphabetical order?

$\boxed{546,000}$

$\binom{26}{3} * \binom{10}{4}$ no permutations allowed

5. True and false questions. Instructions: for each statement below, circle TRUE if the statement is always true and circle FALSE otherwise. Your work will NOT be graded. (5 pts. each)

(i) There are exactly 8 subsets of $\{a, b, c\}$, including the emptyset and the set itself. $\boxed{\text{TRUE}}$ FALSE

True 3 binary choices:

a included (yes/no)?
b included (yes/no)?
c included (yes/no)?

$2^3 = 8$

(ii) The statement $p \leftrightarrow (p \vee \neg p)$ is a tautology. TRUE $\boxed{\text{FALSE}}$

false. when p is false the double-arrow is false

(iii) If $\mathcal{A}$ is a tautology and $\mathcal{B}$ is any statement, then $\mathcal{B} \to \mathcal{A}$ is a tautology. $\boxed{\text{TRUE}}$ FALSE

True. $\mathcal{B} \to \mathcal{A} \Longleftrightarrow \neg\mathcal{B} \vee \mathcal{A}$

and disjunction of tautology w/ any statement is a tautology.

(iv) $2 + 4 + 6 + \cdots + 2n = n(n + 1)$ for all integers $n \geq 1$. $\boxed{\text{TRUE}}$ FALSE

True. $2+4+6+\ldots+2n = 2(1+2+3+\ldots+n) = 2\frac{(n+1)n}{2} = n(n+1)$

(v) There exists an integer n such that for every integer m, if $m \geq n$ then $\lceil \frac{n}{m} \rceil = 1$. $\boxed{\text{TRUE}}$ FALSE

True. All integers $n>0$

This property.

Alternate Solution to #3(a).

$$\begin{array}{c} q \longrightarrow [\neg(p \to r)] \\ \neg(\neg r \wedge s) \\ s \wedge p \\ \hline \neg q \end{array}$$

p	q	r	s	$s \wedge p$	$r \vee \neg s$ $\neg(\neg r \wedge s)$	$\neg q \vee (p \wedge \neg r)$ $\neg q \vee \neg(\neg p \vee r)$ $q \to (\neg(p \to r))$	$\neg q$
T	T	T	T	T	T	F	—
T	T	T	F	F	T	F	—
T	T	F	T	T	F	—	—
T	T	F	F	F	—	—	—
T	F	T	T	T	T	F	—
T	F	T	F	F	—	—	—
T	F	F	T	T	T	T	T
T	F	F	F	F	—	—	—
F	T	T	T	F	—	—	—
F	T	T	F	F	—	—	—
F	T	F	T	F	—	—	—
F	T	F	F	F	—	—	—
F	F	T	T	F	—	—	—
F	F	T	F	F	—	—	—
F	F	F	T	F	—	—	—
F	F	F	F	F	—	—	—

There is ONLY ONE ASSIGNMENT that makes all the assumptions true. Under that assignment, the conclusion is true. Hence, whenever all the assumptions are true the conclusion is also true. So the argument is valid. ∎