Practice Exam 2

1. What can you say about a graph on four vertices that has 6 edges? Generalize this statement.

2. Prove or disprove:

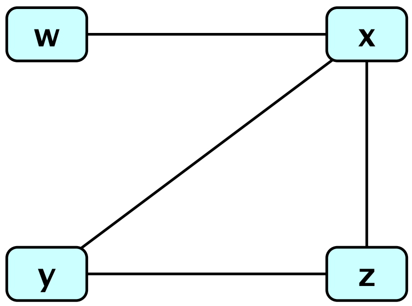
(a) Every connected graph that has at least three vertices of degree 2 contains $\mathcal{K}_3$ as a subgraph.

(b) Every graph that has at least one edge contains $\mathcal{K}_2$ as a subgraph.

(c) The complete graph on four vertices is planar.

(d) The isomorphism classes of graphs with no edges corresponds to the number of vertices in the graph.

3. Consider $\mathcal{G} = (\{a, b, c, d\}, \{ab, bc, bd, cd\})$ and the graph $\mathcal{H}$ whose model is



Which of the following are isomorphisms from $\mathcal{G}$ to $\mathcal{H}$? Explain.

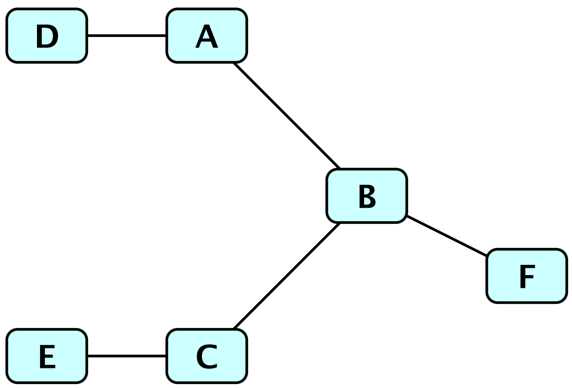
(a)
$$\phi : \{a, b, c, d\} \to \{w, x, y, z\}$$
$$\phi(a) = w,\ \phi(b) = x, \phi(c) = y,\ \phi(d) = z.$$

(b)
$$\phi : \{a, b, c, d\} \to \{w, x, y, z\}$$
$$\phi(a) = w,\ \phi(b) = x, \phi(c) = z,\ \phi(d) = y.$$

(c)
$$\phi : \{a, b, c, d\} \to \{w, x, y, z\}$$
$$\phi(a) = x,\ \phi(b) = w, \phi(c) = y,\ \phi(d) = z.$$

4. Explain the difference between *graph* and *pseudograph*. Is every graph a pseudograph? Is every pseudograph a graph?

5. Consider the graph $\mathcal{G}$ whose model is shown.



(a) Give a description of $\mathcal{G}$ using the definition.

(b) Is $\mathcal{G}$ bipartite? If so, give a bipartition and draw a model which shows that $\mathcal{G}$ is bipartite. If not, explain why $\mathcal{G}$ is not bipartite.

(c) How many subgraphs of $\mathcal{G}$ contain exactly 2 vertices each having degree 1?

(d) How many of these subgraphs are non-isomorphic? In other words, how many isomorphism classes are there for subgraphs of $\mathcal{G}$ that contain exactly 2 vertices of degree 1?

(e) How many subgraphs of $\mathcal{G}$ contain exactly 3 vertices and have degree sequence $3, 1, 1$?

6. Write an algorithm which takes as input n real numbers $a_1, \ldots, a_n$ and outputs the number of times that a_1 occurs in the list. What is the complexity of your algorithm *(be specific)*.

7. Prove that e^n is not $O(n^k)$ for any natural number k.

8. Solve the recurrence relation $a_n = sa_{n-1} + ta_{n-2}$ for various values of s, t which are integers and initial conditions a_0, a_1 which are also integers (usually most of the parameters are non-negative integers such that the characteristic polynomial factors and the linear equations you solve are not *too* nasty).

9. Let $\mathcal{G}_1$ denote the graph on one vertex z. Define inductively $\mathcal{G}_n$ for $n > 1$ as follows: the vertex set of $\mathcal{G}_n$ is the vertex set of $\mathcal{G}_{n-1}$ appended with one more vertex v which is not a vertex of $\mathcal{G}_{n-1}$, and the edge set of $\mathcal{G}_n$ is the edge set of $\mathcal{G}_{n-1}$ appended with the edge (z, v).

(a) Draw a model for $\mathcal{G}_n$ for some small values of n. What would you like to call the model? *(That is, find the obvious name for the graph $\mathcal{G}_n$)*

(b) Find a closed formula for the number of edges in $\mathcal{G}_n$. Find a recursively defined formula.