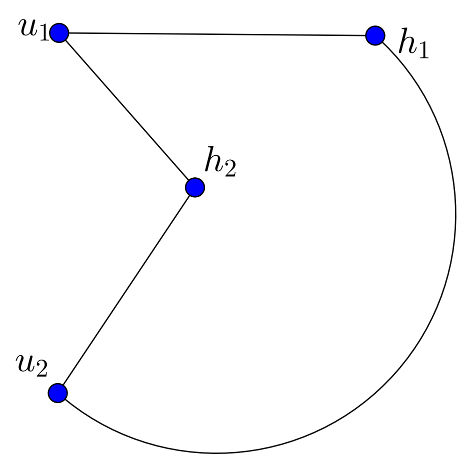
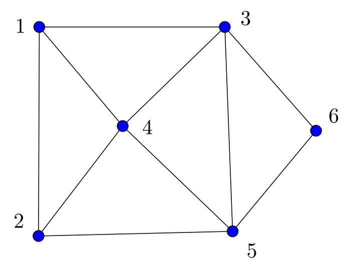
Homework 10: Core solutions

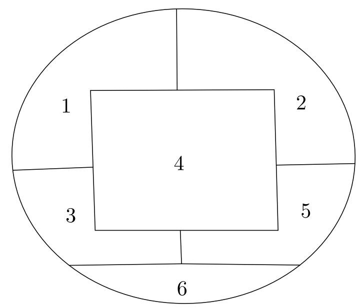
Section 9.1 on page 287 problems 2, 6, 7, 10.
Section 9.2 on page 294 problems 1, 2, 3, 5.

2. *Draw a configuration of two houses and two utilities, each house connected to each utility, but with no crossovers.* *Solution:*



6. *In the graph whose model is shown, the vertices represent the rooms of a one-story house, and an edge between vertices means that the corresponding rooms have a wall in common. Draw a possible floor plan of this house.* *Solution:*





7. *You and a friend meet three other couples at a party and several handshakes take place. Nobody shakes hands with himself or herself, there are no handshakes within couples, and no one shakes hands with the same person twice. The numbers of hands shaken by the other 7 people (excluding you) are all different. How many hands did you shake? How many did your partner shake? Use a graph to aid your solution. Solution:*

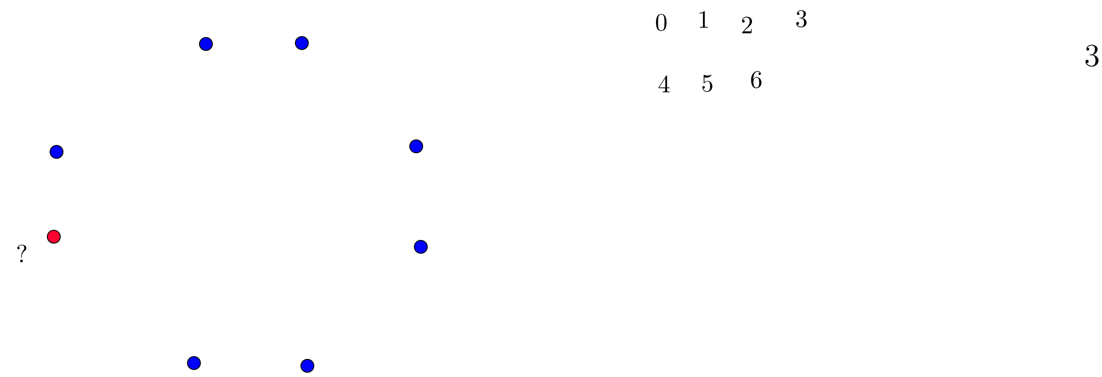
Note that it helps to do the case when there is only 1 other couple first, then the case when there are 2 other couples. Then it is not too hard to see what to do in the case when there are 3 other couples (although it is certainly getting complicated at that point). Here is the idea:

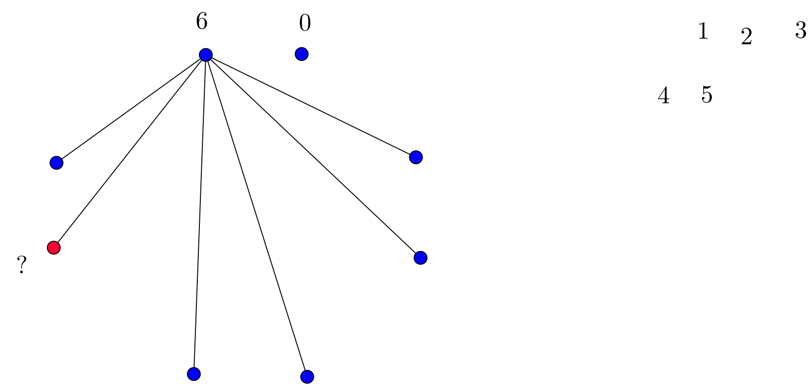
Start with 6 vertices, one for each person. I am the red vertex, and the couples are next to each other. Since the other 7 vertices (people) have different degrees (number of handshakes), there has to be exactly one vertex who has degree 6. The partner of the degree 6 vertex needs to be degree 0. (why?)

Now we are not allowed to connect any more edges to the degree 6 nor degree 0 vertex. We still need a degree 5 vertex, so we choose one of the other pairs and make one degree 5 and the other already has degree 1.

Finishing it off in the same way we get that my partner and I each shake 3 hands.

□


0 1 2 3
4 5 6
?
3


6 0
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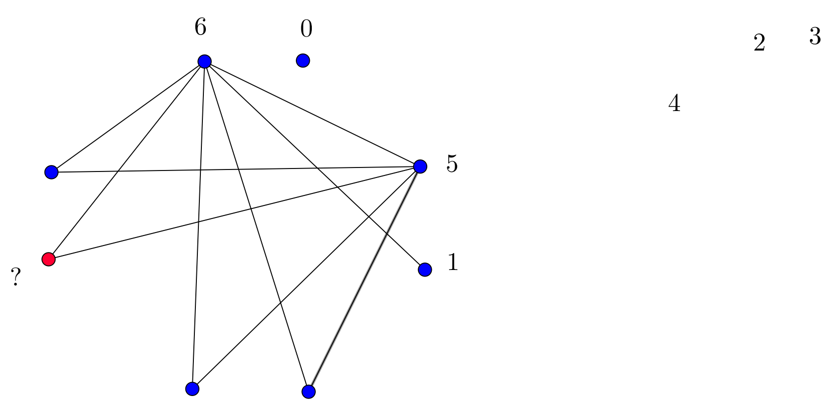
1 2 3
4 5


6 0
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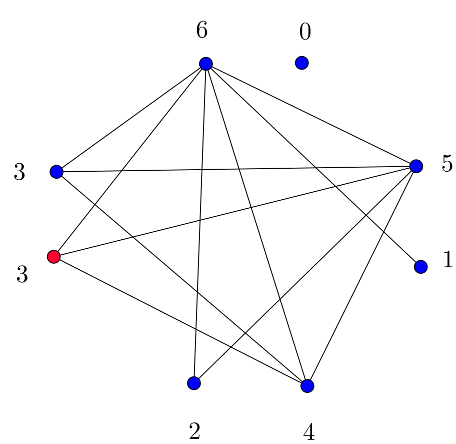
1
2 3
4


6 0
3
5
3
1
2 4

10. *A graph has six vertices, every two of which are joined by an edge. Each edge is colored red or white. Show that the graph contains at least two monochromatic triangles. Solution:* Getting the first monochromatic triangle is easy and we already saw how to do it in Supplementary Homework #5. Since I haven't given the proof yet, I'll provide it.

Pick a vertex, which I will call the "home vertex". By the pigeonhole principle there are at least 3 monochromatic edges touching this vertex, say red ones. If any of the 3 vertices touching the home vertex along these red edges share a red edge amongst themselves, then we have found the first monochromatic triangle (it is red). Otherwise, since none of the 3 vertices touching the home vertex share a red edge, all their edges are white (and we have assumed that the graph is complete, don't forget). So there is our first monochromatic triangle (in this case it is white).

Now, let us call the vertices of the monochromatic triangle we have found "special" and the other vertices "other". For simplicity (after putting on special glasses if necessary) let us say that the monochromatic triangle we found was red.

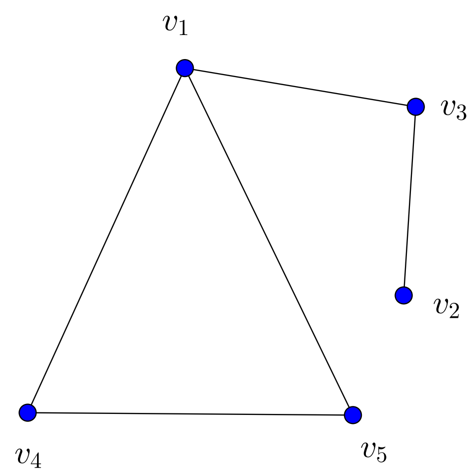
If the three other vertices, those not in the monochromatic triangle we found above, are connected by monochromatic edges, then we are done. Suppose otherwise, so that there is at least one vertex connected to the other two by non-chromatic edges. Call this vertex the new "home" vertex h, and the vertex o_1 which is other and connected to h by a red edge and o_2 the vertex which is other and connected to h by a white edge. If the home vertex is connected to two special vertices by red edges, we are done since this gives us a second red triangle. Suppose otherwise, then there are two edges coming out of home, one of which is red and the other is white. Let s_1 denote the special vertex such that (h, s_1) is red and s_2 denote the special vertex such that (h, s_2) is white. Let s_3 denote the other special vertex.

Now, if the edge (o_2, s_2) is white, then we are done because h, o_2, s_2 would form a white triangle. Suppose otherwise, so (o_2, s_2) is red.

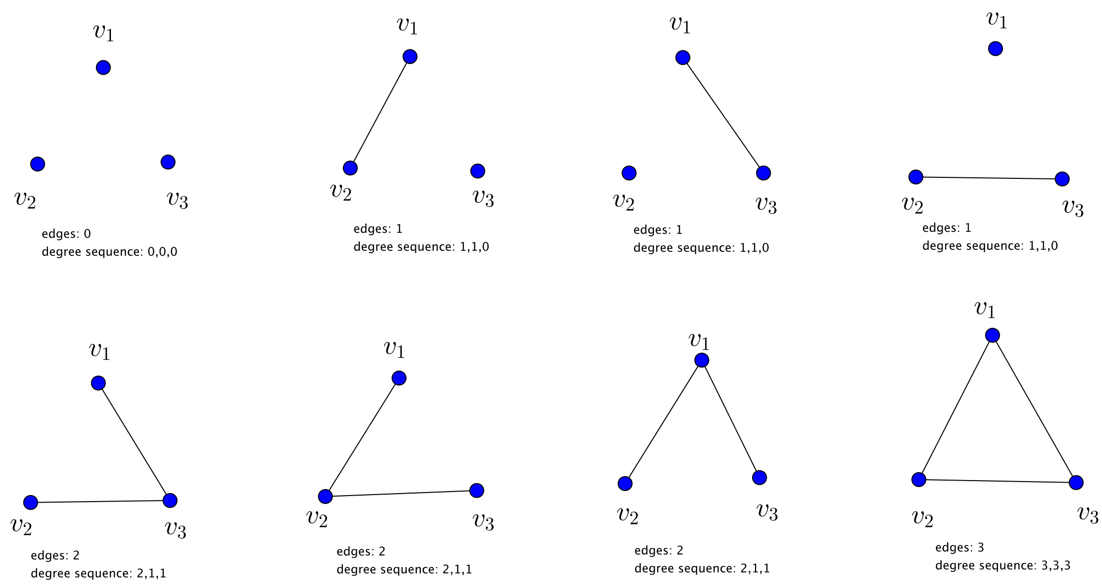
Finally, if (o_2, s_2) is either red or white we have found our second monochromatic triangle (it is the triangle formed by h, o_2, s_3 if the edge is white and the triangle formed by s_2, o_2, s_3 if the edge is red).

$\square$

1. *Draw a model for a graph with five vertices v_1, v_2, v_3, v_4, v_5 such that* $\deg(v_1) = 3$*, v_2 is an odd vertex,* $\deg(v_3) = 2$*, and v_4 and v_5 are adjacent. Solution:*



v_1
v_3
v_2
v_4
v_5



v_1 v_2 v_3
edges: 0
degree sequence: 0,0,0
v_1 v_2 v_3
edges: 1
degree sequence: 1,1,0
v_1 v_2 v_3
edges: 1
degree sequence: 1,1,0
v_1 v_2 v_3
edges: 1
degree sequence: 1,1,0
v_1 v_2 v_3
edges: 2
degree sequence: 2,1,1
v_1 v_2 v_3
edges: 2
degree sequence: 2,1,1
v_1 v_2 v_3
edges: 2
degree sequence: 2,1,1
v_1 v_2 v_3
edges: 3
degree sequence: 3,3,3

2. *Draw models for all possible graphs with three vertices v_1, v_2, v_3. How many edges in each graph? What is the degree sequence of each graph? Does this question make sense for pseudographs?* *Solution:* We expect $2^{\binom{n}{2}} = 8$ graphs when $n = 3$. The question has no concrete answer for pseudographs since any number of edges are possible, although we could probably say something about the degree sequences that were possible.

3. *Give an example of a graph such that every vertex is adjacent to two vertices and every edge is adjacent to two edges.* *Solution:* The square works:

$$\mathcal{G} = (\{a, b, c, d\}, \{ab, bc, cd, da\}).$$

5. *Give an example of a graph with five vertices and as many edges as possible. How many edges does your graph contain? What is the name of this graph and how is it denoted?* *Solution:* The graph $\mathcal{K}_5$ is called the complete graph on 5 vertices and has the property that it has all the edges that are possible, of which there are a total of $\binom{n}{2} = 10$ when $n = 5$.