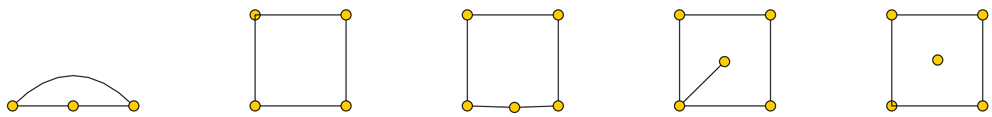
Math 2602 — Finite and Linear Math — Fall '14

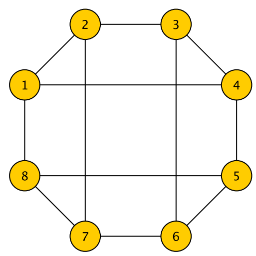
Homework 11: Core solutions

Section 9.2 on page 295: 14(all), 15a-b, 18ade, 20b, 22b, 23a, 29.

Section 9.3 on page 299 problems 1all, 3all, 4a, 4b, 4c.

14. *Which of the graphs depicted is a subgraph of the graph depicted below?*

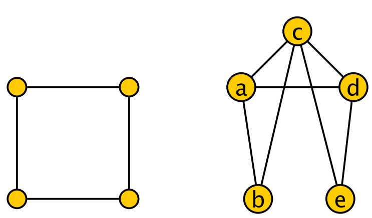




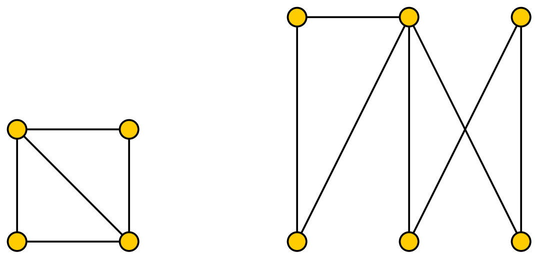
Solution: Let the large model denote the graph $\mathcal{G}$. Let the first five graphs depicted be called graphs (i), (ii), (iii), (iv), and (v). Graph (i) is not a subgraph of $\mathcal{G}$ since (i) is a cycle of length 3, and $\mathcal{G}$ is bipartite (see below). Graph (ii) is clearly a subgraph, for example the vertices of (ii) could be 2,3,6,7 reading clockwise. Graph (iii) is clearly not a subgraph since it is a cycle of odd length, hence not contained in $\mathcal{G}$ (again, see below). Graph (iv) and graph (v) are clearly subgraphs of $\mathcal{G}$ with outer vertices 2,3,6,7 and middle vertex 5 and outer vertices 2,3,4,1 and middle vertex 6, respectively.

□

15. *For each pair of graphs discover whether the graph on the left is a subgraph of the one on the right. If not, explain. If it is, give a description which verifies your assertion.*



(a) *Solution:* The model on the left is of the graph $\mathcal{G} = (\{a, b, c, d\}, \{ab, bc, cd, da\})$ and the model on the right is of the graph $\mathcal{H} = (\{a, b, c, d, e\}, \{ab, ac, ad, bc, cd, ce, de\}$. Clearly $\mathcal{G} \subseteq \mathcal{H}$.

(b)  *Solution:* The graph $\mathcal{G}$ depicted on the left has two vertices of degree 3, but the graph $\mathcal{H}$ depicted on the right only has one vertex of degree 3, hence $\mathcal{G}$ is not a subgraph of $\mathcal{H}$.

□

18. *Determine if there is a graph with the given degree sequence.*

- **(a)** $4, 4, 4, 3, 2$ *Solution:* No. The sum of the degrees in the degree sequence must be even.
- **(d)** $1, 1, 1, 1, 1, 1$ *Solution:* Yes. The graph $\mathcal{G} = (\{a, b, c, d, e, f\}, \{ab, cd, ef\})$ has this degree sequence.
- **(e)** $5, 4, 3, 2, 1$ *Solution:* No. The sum of the degrees in the degree sequence must be even.

20b. *If a graph has degree sequence* $5, 5, 4, 4, 3, 3, 3, 3$*, how many edges does it have? Solution:* Since

$$\sum_{v \in V} \deg(v) = 2|E|$$

and the sum of the degrees in the degree sequence is

$$5 + 5 + 4 + 4 + 3 + 3 + 3 + 3 = 30$$

there must be 15 edges in the graph.

□

22b. *Show that any graph that contains a an n-cycle, with n odd is not bipartite. Solution:* Suppose, seeking a contradiction that there is a bipartite graph which contains an n-cycle for some odd number n. Then, there exists a bipartition V_1, V_2 of the vertex set. So $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$ (V_1, V_2 are a partition of V) and furthermore every edge of the graph takes the form (v_1, v_2) for some $v_1 \in V_1$ and some $v_2 \in V_2$.

Denote the n-cycle by $a_1 a_2 \cdots a_n a_1$. Suppose a_1 is in V_1. Then a_2 is in V_2, and hence a_3 must be in V_1 again. Continuing on we see that $a_i \in V_1$ if i is odd and $a_i \in V_2$ if i is even. But since n is odd we have that $a_n \in V_1$. This is a problem since this means that the edge (a_n, a_1) connects two vertices of V_1. This is a contradiction, so no bipartite graph can contain an odd cycle.

□

23a. *Must a subgraph of a bipartite graph be bipartite? Solution:* Yes. Here is the proof:

Let $\mathcal{G} = (V, E)$ be a bipartite graph. Let $\mathcal{H}$ be any subgraph of $\mathcal{G}$, so

$$\mathcal{H} = \left(\widetilde{V}, \widetilde{E}\right)$$

for some $\widetilde{V} \subseteq V$ and $\widetilde{E} \subseteq E$. Let V_1, V_2 be bipartition sets of $\mathcal{G}$. So

$$\begin{aligned} & V_1 \cup V_2 = V, \\ & V_1 \cap V_2 = \emptyset, \text{ and} \\ \text{for all } (v, w) \in E \quad & (v \in V_1 \to w \in V_2) \wedge (v \in V_2 \to w \in V_1). \end{aligned}$$

We claim that $\widetilde{V} \cap V_1, \widetilde{V} \cap V_2$ is a bipartition of $\widetilde{V}$. Clearly, these two sets are a partition of $\widetilde{V}$ (they are disjoint and their union is all of $\widetilde{V}$). We just have to show that the edges of $\mathcal{H}$ connect a vertex of $\widetilde{V} \cap V_1$ with a vertex of $\widetilde{V} \cap V_2$.

Let e be one of the edges in $\mathcal{H}$. Since it is also an edge of $\mathcal{G}$ it connects a vertex $v \in V_1$ to a vertex $w \in V_2$. So $e = (v, w)$. Since e was an edge of $\mathcal{H}$, both vertices v, w must be in $\widetilde{V}$ (since $\mathcal{H}$ is a graph). So $v \in \widetilde{V} \cap V_1$ and $w \in \widetilde{V} \cap V_2$.

□

29. *Let m and M denote the minimum and maximum degrees of the vertices of a graph $\mathcal{G} = (V, E)$. Show that*

$$m \le \frac{2|E|}{|V|} \le M.$$

Solution: This is immediate from the equality of Euler

$$\sum_{v \in V} \deg(v) = 2|E|.$$

In particular,

$$\sum_{v \in V} \deg(v) \leq \sum_{v \in V} (\max_{v \in V} \deg(v)) = |V| \cdot \max_{v \in V} \deg(v),$$

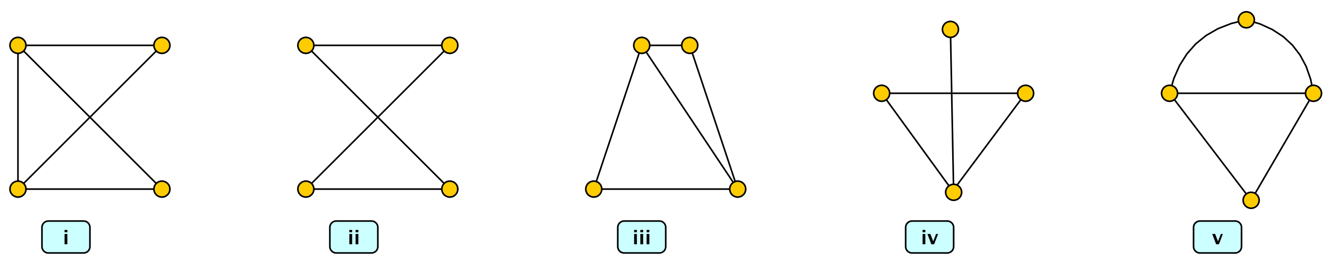
and dividing both sides of the inequality by $|V|$ (the question makes no sense anyway when $V = 0$) we obtain

$$\frac{1}{|V|} \sum_{v \in V} \deg(V) \leq \max_{v \in V} \deg(v).$$

After substituting the sum on the left hand side of the above equality with $2|E|$ using Euler's equality we have the desired result. The case for the minimum is similar.

□

1. *For each of the ten pairs of graphs obtained from those shown, exhibit an isomorphism or explain why one does not exist.*



Solution: The only one which is not obvious is that (i) and (iii) are isomorphic. See table below for justifications for the other 9 pairs.

	(ii)	(iii)	(iv)	(v)
(i)	No. (ii) does not have a vertex of degree 3.	Yes*.	No. Only (iv) has a vertex of degree 1.	Yes**.
(ii)		No. (ii) does not have a vertex of degree 3.	No. Only (iv) has a vertex of degree 1.	No. (ii) does not have a vertex of degree 3.
(iii)			No. Only (iv) has a vertex of degree 1.	Yes. This is obvious.
(iv)				No. Only (iv) has a vertex of degree 1.

** Note that the pair (i) and (v) are isomorphic since (i) and (iii) are isomorphic (see below) and (iii) and (v) are clearly isomorphic (and isomorphism is a transitive property).

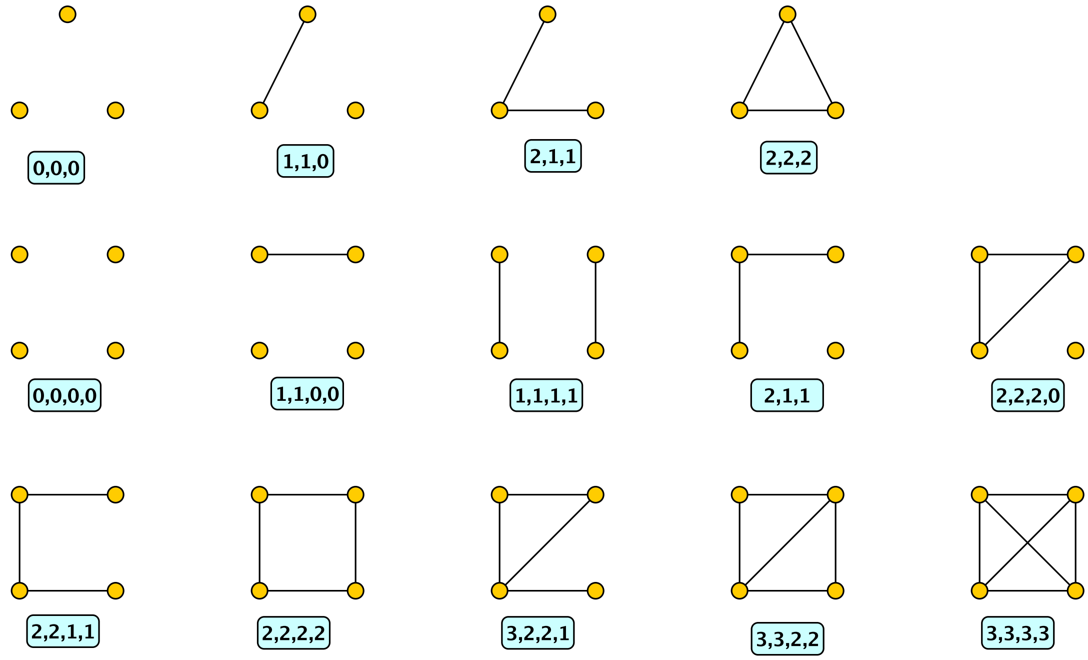
* To see (i) is isomorphic to (iii): Label the vertices of (i) by a, b, c, d starting from the top left of the model above and proceeding clockwise. Label the vertices of (iii) by v, w, x, y in the same manner. Then $\phi : \{a, b, c, d\} \to \{v, w, x, y\}$ defined by

$$\begin{aligned} a &\mapsto v \\ b &\mapsto w \\ c &\mapsto y \\ d &\mapsto x \end{aligned}$$

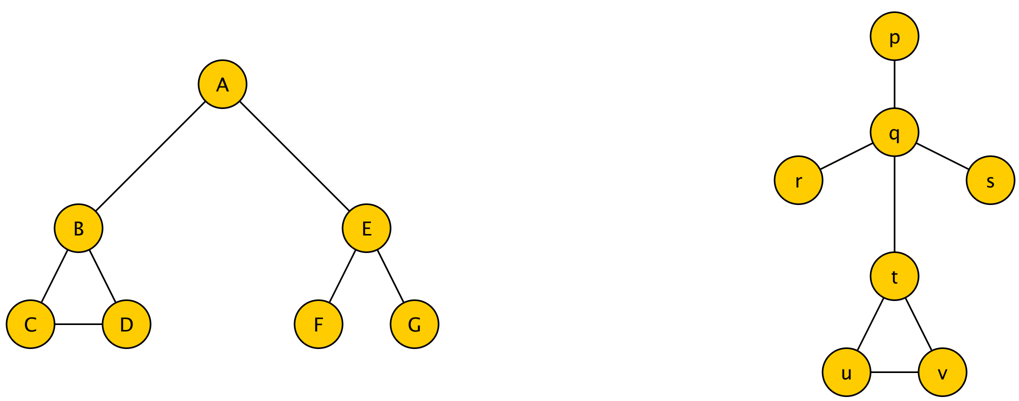
is an isomorphism (it is clearly bijective and it is easy to check that it is edge preserving).

□

3. *Draw models for all non-isomorphic graphs on* $n = 3$ *and* $n = 4$ *vertices. Give the degree sequence of each. Solution:*

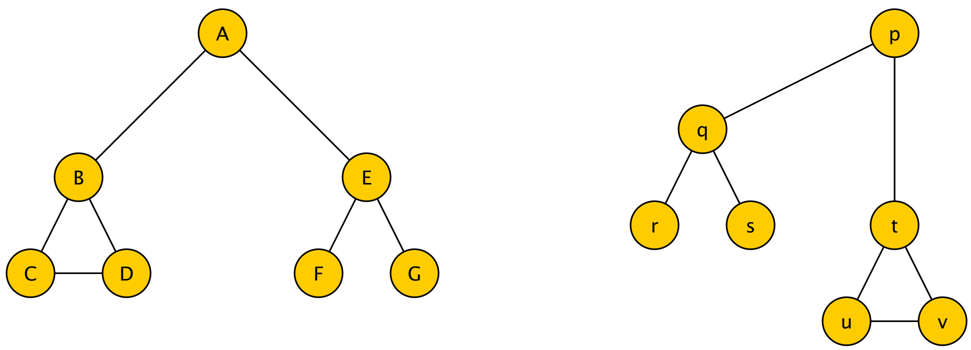


4. *For each pair explain why the graphs are not isomorphic or give an isomorphism.*



(a)

Solution: No. The vertex q in the model on the right has degree 4 while no vertex in the model on the left has degree 4.



(b)

Solution: Yes. Clearly the map $\phi : \{A, B, C, D, E, F, G\} \to \{p, q, r, s, t, u, v\}$ defined by $\phi(A) = p,\ \phi(B) = t,\ \phi(C) = u,\ \phi(D) = v,\ \phi(E) = q,\ \phi(F) = r,\ \phi(G) = s$ is such an isomorphism.

Extra problem: *Discuss isomorphisms of pseudographs. Solution:* A pseudograph on one vertex is determined, up to the label on the vertex, by how many loops there are at the vertex. Thus, there are as many pseudographs, up to isomorphism, as there are integers which are non-negative.

A psedograph on two vertices is determined, up to the labeling, by how many loops at each vertex and how many multiple edges there are between the vertices. Thus, denoting $\mathbb{N}$ the non-negative integers we have that there is a bijection between the isomorphism classes of pseudographs on two vertices and $\mathbb{N}^3$.

In any pseudograph, any degree sequence on the vertices is possible. This is seen, for example, by putting the right number of loops at each vertex to get the desired degree sequence. More specifically, if the degree sequence you want to achieve is $d_1, \ldots, d_n$ on vertices $v_1, \ldots, v_n$, then putting d_i loops on vertex d_i produces a pseudograph with the desired degree sequence. Of course, there are many other non-isomorphic pseudographs which give the same degree sequence (for all but a few degree sequences). □