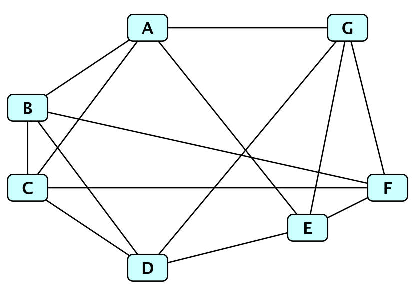
Homework 12: Core problems

Section 10.1 on page 309: 1, 3, 4bf, 11, 12, 13, 15, 16, 17, 18 , 21, 23, & Extra problem.

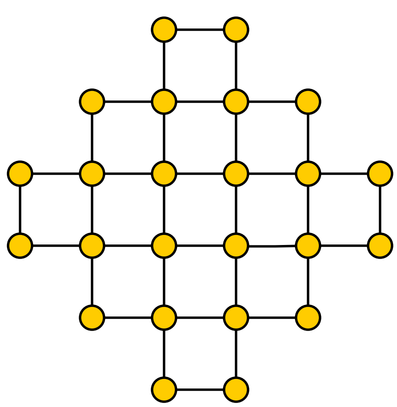
1a. *Find a connected graph with as few vertices as possible that has precisely two vertices of odd degree.*

1b. *Find a connected graph with as few vertices as possible that has precisely two vertices of even degree.*

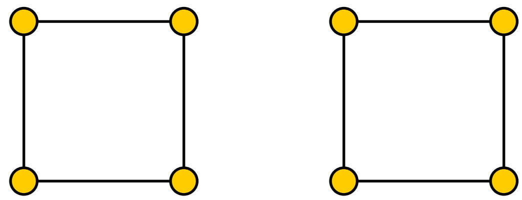
3. *In each case, explain why the graph is Eulerian and find a Eulerian circuit.* ***ditto prob 4bf***



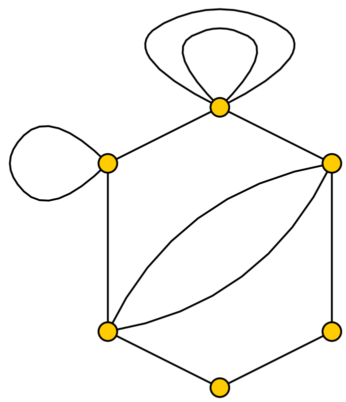
(a)



(b)



4b.



4f.

11. *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are Eulerian graphs with no vertices in common. Let $v_1 \in V(\mathcal{G}_1)$ and $v_2 \in V(\mathcal{G}_2)$, and consider the graph $\mathcal{H}$ obtained from $\mathcal{G}_1, \mathcal{G}_2$ by connecting v_1 and v_2 by an edge,*

$$\mathcal{H} = (V(\mathcal{G}_1) \cup V(\mathcal{G}_2), E(\mathcal{G}_1) \cup E(\mathcal{G}_2) \cup \{(v_1, v_2)\}).$$

Is $\mathcal{H}$ Eulerian? Prove your answer.

12. *For what values of $n > 1$ is $\mathcal{K}_n$ Eulerian? Prove your answer. For what values of $n > 1$ does $\mathcal{K}_n$ posses a Eulerian trail (that is, a trail from two vertices of $\mathcal{K}_n$ that passes through every vertex and contains every edge).*

13. *The graph $\mathcal{K}_{m,n}$ is called the bipartite complete graph on (m, n). It is a bipartite graph with vertices $\{v_1, \ldots, v_m, w_1, \ldots, w_n\}$ and edge set $\{(v_i, w_j)|\ 1 \leq i \leq m,\ 1 \leq j \leq n\}$ (do you know how many edges it has?). Find a necessary and sufficient condition on natural numbers m, n such that $\mathcal{K}_{m,n}$ is Eulerian, and prove your answer. Then, do the same for having a Eulerian trail.*

15. *Prove that any circuit in a graph must contain a cycle, and that any circuit which is not a cycle contains at least two cycles.*

16. *Is it true that any closed walk in a graph contains a cycle? Prove or give a counterexample.*

17. *Let v, w be two distinct vertices of a graph $\mathcal{G}$. Prove that there exists a walk from v to w if and only if there exists a path from v to w.*

18. *For any two vertices $v, w \in V(\mathcal{G})$ write $v \sim w$ if there is a walk from v to w or if $v = w$. Prove that this gives an equivalence relation on $V(\mathcal{G})$ (that is, a relation that is reflexive, symmetric, and transitive). The equivalence classes are herby called* ***connected components of*** $\mathcal{G}$*. Why is this a good name?*

21. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two graphs. Prove that if $\mathcal{G}_1 \simeq \mathcal{G}_2$ are isomorphic then either they are both connected or they are both disconnected. Does the converse of this statement hold?*

23. *Let $\mathcal{G}$ be a connected graph with $n > 1$ vertices. Prove that $\mathcal{G}$ has at least $n - 1$ edges, and that if no vertex has degree 1, then $\mathcal{G}$ has at least n edges.*

Extra problem. *Show that the book's definition of connectedness agrees with the definition from class. That is, show that the two definitions below are logically equivalent.*

Definition 1. A graph $\mathcal{G} = (V, E)$ is disconnected if there exist non-empty subgraphs $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ such that V_1 and V_2 partition V and E_1 and E_2 partition E. A graph is connected if it is not disconnected.

Definition 2. A graph $\mathcal{G}$ is connected if for any two vertices v, w there is a walk between v and w.