Homework 2: Core solutions

These homework solutions have been created by Sal, so let him know if you find any mistakes. The problems assigned in the core section were:

Section 0.1 on page 10 problems 6b-6c, 6f, 6i, 7d, 7f, 7k.

Section 0.2 on page 16 problems 3b, 3d, 3f, 6, 7, 13.

Review Exercises for Chapter 0 on page 17-18 problems 2a, 3a, 15, 16.

Section 1.1 on page 23 problems 1b, 2a, 4.

Questions are in *italics*, and have generally been shortened or reworded, and after long solutions there is a $\square$.

6. *Write down the converse and contrapositive of each implication.* Recall the converse of $p \to q$ is $q \to p$, and the contrapositive of $p \to q$ is $\neg q \to \neg p$.

(b) $x^2 = 1 \to x = \pm 1$. The converse is $x \pm 1 \to x^2 = 1$, while the contrapositive is $x \neq \pm 1 \to x^2 \neq 1$.

(c) *If* $x^2 = x + 1$, *then* $x = 1 + \sqrt{5}$ *or* $x = 1 - \sqrt{5}$. The converse is: If $x = 1 + \sqrt{5}$ or $x = 1 - \sqrt{5}$, then $x^2 = x + 1$. The contrapositive is: If $x \neq 1 \pm \sqrt{5}$, then $x^2 \neq x + 1$.

(f) $ab = 0 \to a = 0$ *or* $b = 0$. The converse is $a = 0$ or $b = 0 \to ab = 0$, while the contrapositive is $a \neq 0$ and $b \neq 0 \to ab \neq 0$.

(i) *If $p(x)$ is a polynomial of odd degree, then $p(x)$ has at least one real root.* The converse is: If $p(x)$ is a polynomial which has at least one real root, then $p(x)$ has odd degree. The contrapositive is: If $p(x)$ is a polynomial which has no real roots, then the degree of $p(x)$ is even.

7. *Rewrite using “for all” and “for some”.*

(d) *There are infinitely many primes.* For every natural number n, there are more than n primes.

(f) *All positive real numbers have real square roots.* For every real number x, if x is positive then the positive and negative square root of x are real numbers.

(k) *Every polynomial of degree 3 has a real root.* For every polynomial $p(x)$, if $p(x)$ has degree 3, then $p(x)$ has a real root.

3. *Exhibit a counter-example.*

(b) *For integers a, b, we have $ab = 1 \to a = b = 1$.* Set $a = b = -1$. Then $ab = 1$ but $a \neq 1$.

(d) *If the average of 4 different integers is 10, then at least one of the numbers must be greater than 12.* The average of 8, 9, 11, and 12 is 10, but none of these numbers is *greater than* 12.

(f) *For all real numbers x, y, we have $x - y = y - x$.* If $x \neq y$ this statement is false. In particular, set $x = 0$ and $y = 1$, then $x - y = -1$ and $y - x = 1$ and $-1 \neq 1$.

6. *State the converse of the following implication and state whether the converse is true or false.*

$$x \text{ is an even integer} \to x + 2 \text{ is an even integer.}$$

The converse is the statement: If $x + 2$ is an even integer, then x is an even integer. Both the statement and it's converse are true.

□

7. *Determine whether the statement in Problem #6 is true or false if $\to$ is replaced with $\leftrightarrow$.* Since the implication and its converse are both true, the two statements are equivalent and so "x is an even integer $\leftrightarrow$ $x + 2$ is an even integer" is a true statement.

□

13. *Consider the following assertions.*

$\mathcal{A}$: There exists a real number y such that $y > x$ for every real number x,

$\mathcal{B}$:For every real number x, there exists a real number y such that $y > x$.

Which one is true and which one is false. Explain carefully and justify your reasoning. The first statement is false. There does not exists a real number y that is greater than all other real numbers. In particular, if y is real then $y + 1$ is real and $y < y + 1$.

The second statement is true. For every real number x there *does* exist a real number y which is bigger than x; in particular set $y = x + 1$. Note, there are actually infinitely many such y's that will work.

□

1b. *Construct a truth table for the compound statement $(p \wedge q) \vee ((\neg p) \to q)$.* This is easy I bet you can do it.

2a. *If $p \to q$ is false, determine the truth value of $(p \wedge (\neg q)) \vee ((\neg p) \to q)$.* If $p \to q$ is false, then p is true but q is false. So, in particular $p \wedge (\neg q)$ is true, and hence the compound statement is true since the first part of the disjunction is true.

4. *Determine the truth value of $[p \to (q \wedge (\neg r))] \vee [r \leftrightarrow ((\neg s) \vee q)]$ in the case where each of $p, q, r,$ and s are false.* When all atomics are false, the first part of the disjunction is true, since it is an implication. Hence the compound statement is true.