

Homework 3: Core solutions

These homework solutions have been created by Sal, so let him know if you find any mistakes. The problems assigned in the core section were:

Section 1.1 on page 23 problems 6a, 9b, 10a.

Section 1.2 on page 29-30 problems 2b, 3b, 4b, 7a, 10c, 10e.

Section 1.3 on page 24 problem 1c-1f.

Questions are in *italics*, and have generally been shortened or reworded, and after long solutions there is a \square .

- 6b.** *Show that $q \rightarrow (p \rightarrow q)$ is a tautology.* We make a truth table for the statement and see that all 4 truth values are T.

p	q	$p \rightarrow q$	$q \rightarrow (p \rightarrow q)$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

- 9b.** *Given that the compound statement \mathcal{A} is a contradiction, establish that if \mathcal{B} is a tautology, then $\mathcal{B} \rightarrow \mathcal{A}$ is a contradiction.* If \mathcal{B} is a tautology, then in any assignment of truth values to the atomic statements involved in \mathcal{B} the truth value of \mathcal{B} is always T. Similarly, the truth value of \mathcal{A} is always F for all assignments of the atomic statements appearing in \mathcal{A} . Therefore, the truth value of $\mathcal{B} \rightarrow \mathcal{A}$ is always F, since the hypothesis is always true and the conclusion is always false.

\square

- 10a.** *Show that the statement $p \rightarrow (q \rightarrow r)$ is **not** logically equivalent to the statement $(p \rightarrow q) \rightarrow r$.* We could make a truth table for each statement and notice that they do not have identical truth values for all assignments $p, q, r \in \{T, F\}$. However, it suffices to find one such assignment where the two statements have different truth values. Note that if p is false and r is true then the first statement is true, but the second statement is false, and that this shows that the two statements are not logically equivalent.

\square

- 2b.** *What is the negation of the statement $p \vee [\neg(p \wedge q)]$? Show that this negation is a contradiction.* The negation of the statement $p \vee [\neg(p \wedge q)]$ is

$$\begin{aligned} \neg[p \vee [\neg(p \wedge q)]] &\iff \neg p \wedge \neg[\neg(p \wedge q)] \\ &\iff \neg p \wedge (p \wedge q) \\ &\iff (\neg p \wedge p) \wedge q. \end{aligned}$$

Now note that the first part of the statement $(\neg p \wedge p) \wedge q$ is a contradiction, and the conjunction of a contradiction with any statement is a contradiction.

□

- 3b.** Simplify the statement $(p \vee r) \rightarrow [(q \vee (\neg r)) \rightarrow ((\neg p) \rightarrow r)]$. We use that for any statements \mathcal{A} and \mathcal{B} the implication $\mathcal{A} \rightarrow \mathcal{B}$ is logically equivalent to $\mathcal{B} \vee \neg \mathcal{A}$. We have,

$$\begin{aligned} (p \wedge r) \rightarrow [(q \vee (\neg r)) \rightarrow ((\neg p) \rightarrow r)] &\iff [(q \vee (\neg r)) \rightarrow ((\neg p) \rightarrow r)] \vee \neg(p \vee r) \\ &\iff ((\neg p) \rightarrow r) \wedge \neg(q \vee (\neg r)) \wedge \neg(p \vee r) \\ &\iff (r \wedge \neg(\neg p)) \wedge \neg(q \vee (\neg r)) \wedge \neg(p \vee r) \\ &\iff (r \wedge p) \wedge (\neg q \wedge r) \wedge (\neg p \wedge r) \\ &\iff p \wedge (\neg p) \wedge (\neg q) \wedge r \end{aligned}$$

Clearly, the statement is a contradiction since $p \wedge (\neg p)$ is a contradiction.

□

- 4b.** Using truth tables, verify the following ABSORPTION property: $(p \vee (p \wedge q)) \iff p$. We make a truth table and check that the truth values of $p \vee (p \wedge q)$ are identical to those of p .

p	q	$p \vee (p \wedge q)$
T	T	T
T	F	T
F	T	F
F	F	F

Since the statement $(p \vee (p \wedge q))$ is true when p is true, and it is false when p is false, this statement is logically equivalent to the statement p .

□

- 7a.** Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are statements with \mathcal{A} logically equivalent to \mathcal{B} . Show that $\mathcal{A} \vee \mathcal{C}$ is logically equivalent to $\mathcal{B} \vee \mathcal{C}$. If \mathcal{A} is logically equivalent to \mathcal{B} , then whenever one is true then so is the other and visa-versa. In particular, if $\mathcal{A} \vee \mathcal{C}$ is false then both \mathcal{A} and \mathcal{C} are false, and in particular \mathcal{B} is false (since \mathcal{A} is!) and thus the statement $\mathcal{B} \wedge \mathcal{C}$ is false. However, if $\mathcal{A} \wedge \mathcal{C}$ is true, then either \mathcal{A} is true or \mathcal{C} is true. In the former case, \mathcal{B} is also true (since it is logically equivalent to \mathcal{A}) and hence $\mathcal{B} \vee \mathcal{C}$ is true. In the latter case, \mathcal{C} is true so the statement $\mathcal{B} \vee \mathcal{C}$ is true as well, regardless of the truth value of \mathcal{B} . Since $\mathcal{A} \wedge \mathcal{C}$ and $\mathcal{B} \wedge \mathcal{C}$ take the same truth values in all cases, these two statements are logically equivalent. □

10. *Express the following in disjunctive normal form:* Recall that the disjunctive normal form of a statement consists of a disjunction of conjunctions. (it is an OR of ANDs)

(c) $p \rightarrow q$. Since $p \rightarrow q$ is logically equivalent to $q \vee \neg p$, and $q \vee \neg p$ is in disjunctive normal form, this is the disjunctive normal form of $p \rightarrow q$.

(e) $(p \vee q) \wedge ((\neg p) \vee (\neg q))$. We use De Morgan's laws.

$$\begin{aligned} (p \vee q) \wedge ((\neg p) \vee (\neg q)) &\iff [(p \vee q) \wedge (\neg p)] \vee [(p \vee q) \wedge (\neg q)] \\ &\iff q \vee p. \end{aligned}$$

Note that since the statement in the last line is in disjunctive normal form and it is logically equivalent to the original statement, this is the disjunctive normal form of the original statement.

□

1. *Determine whether or not each of the following arguments is valid.*

$$p \rightarrow q$$

(c) $\frac{r \rightarrow q}{r \rightarrow p}$. **Invalid.** If p is false and both r and q are true then the assumptions are true but the conclusion is false.

$$p \rightarrow q$$

(d) $\frac{(q \vee (\neg r)) \rightarrow (p \wedge s)}{s \rightarrow (r \vee q)}$. **Valid.** First, we rewrite $p \rightarrow q$ as $q \vee (\neg p)$ and rewrite $(q \vee (\neg r)) \rightarrow (p \wedge s)$ as $(p \wedge s) \vee \neg(q \vee (\neg r))$, then we form the conjunction of the two rewritten assumptions and simplify.

$$\begin{aligned} [q \vee (\neg p)] \wedge [(p \wedge s) \vee \neg(q \vee (\neg r))] &\iff [(q \vee (\neg p)) \wedge (p \wedge s)] \vee [(q \vee (\neg p)) \wedge \neg(q \vee (\neg r))] \\ &\iff [q \wedge p \wedge s] \vee [(q \vee (\neg p)) \wedge ((\neg q) \wedge r)] \\ &\iff [q \wedge p \wedge s] \vee [(\neg q) \wedge (\neg p) \wedge r] \end{aligned}$$

Note that we have found the disjunctive normal form of the conjunction of the two assumptions of this problem. Now, we are asked if the conclusion follows from the assumptions. The conclusion $s \rightarrow (r \vee q)$ can be rewritten as (*i.e.*, is logically equivalent to) $r \vee q \vee (\neg s)$. If we assume that the assumptions are true, then there are two cases to consider. If q is true, then the conclusion is true. If q is false, then r is true (cf. the disjunctive normal form above), and hence the conclusion is true. In either case, the conclusion is true so the argument is **valid**. □

- (e)
$$\frac{p \rightarrow (\neg q) \quad r \rightarrow q}{r} \quad \text{Valid.}$$
 If the assumptions are true, then the third assumption asserts

$\neg p$ that r is true. Since r is true, the second assumption asserts that q is true. Since the first assumption is true, the implication $p \rightarrow (\neg q)$ is true, and since q is true this means that p must be false (or else $p \rightarrow (\neg q)$ would be of the form $T \rightarrow F$, which would be a contradiction with the fact that the implication was assumed to be true). Since we have concluded that $\neg p$ is true assuming that the assumptions are true, the argument is **valid**.

□

- (f)
$$\frac{p \rightarrow (\neg q) \quad (\neg r) \rightarrow p}{r} \quad \text{Valid.}$$
 Since q is true, p must be false or else the implication

$p \rightarrow (\neg q)$ would be false. Since p is false, $\neg r$ must be false or else the implication $(\neg r) \rightarrow p$ would be false. Finally, since $\neg r$ is false we have that r is true.

□