Homework 4: Core solutions

Section 1.3 on page 34-35 problems 3b, 4g, 5i, 5j.

Review Exercises for Chapter 1 on page 36-37 problems 2, 3b, 3d, 5a, 8a, 10a, 10b.

Section 5.1 on page 156-157 problems 3, 4d, 4k, 5a, 13.

3b. *Verify that the argument is valid.*

$$\begin{array}{c} p \to r \\ q \to s \\ \hline (p \wedge q) \to (r \wedge s) \end{array}$$

If we simplify the assumptions and then form their conjunction, we obtain $(r \vee \neg p) \wedge (s \vee \neg q)$. Simplifying further, using De Morgan's law we obtain $[(r \vee \neg p) \wedge s] \vee [(r \vee \neg p) \wedge \neg q]$, and finally we see that the conjunction of the assumptions is logically equivalent to

$$(r \wedge s) \vee (\neg p \wedge s) \vee (r \wedge \neg q) \vee (\neg p \wedge \neg q)$$

Note that this is the disjunctive normal form of the conjunction of the assumptions. In order to verify the validity of the argument, it suffices to assume that both p and q are true, since in all other cases the conclusion is true. If p and q are both true then looking at the disjunctive normal form above, we see that in order for this statement to be true it must be the case that $r \wedge s$ is true, since all the other statements in the disjunction are false. Hence, we have concluded that $r \wedge s$ is true, so the conclusion $(p \wedge q) \to (r \wedge s)$ is true, as desired.

□

4g. *Determine if the argument is valid or invalid.*

$$\begin{array}{l} p \vee (q \to r) \\ q \vee r \\ r \to p \\ \hline p \end{array}$$

The first assumption is logically equivalent to $p \vee r \vee \neg q$, and the third assumption is logically equivalent to $p \vee \neg r$. Therefore, if all assumptions are true we have that the following statement is true; note that it is in *conjunctive normal form*, and it is equivalent to the conjunction of the three assumptions.

$$(p \vee r \vee \neg q) \wedge (q \vee r) \wedge (p \vee \neg r).$$

Assume, seeking a contradiction, that p is false. Then either r or $\neg q$ is true from the first part of the statement above. But $\neg r$ is true, since p is false by the third part of the statement above. Hence, $\neg q$ is true and $\neg r$ is true. Therefore, the statement above is **false**. This is a contradiction with the assumption that this statement was true, so p can not be false. This completes the proof. □

5. *Determine the validity of the argument.*

(h) *If I work hard, then I earn lots of money.*
If I earn lots of money, then I pay high taxes.
___
If I do not work hard, then I do not pay high taxes.

Invalid. Let p be the statement "I work hard", q be the statement "I earn lots of money", and r be the statement "I pay high taxes". Then the argument can be written as

$$\begin{array}{c} p \to q \\ q \to r \\ \hline \neg p \to \neg r. \end{array}$$

If p and q are false but r is true, then both the assumptions are true but the conclusion is false. Hence, the argument is invalid. Note that in terms of the original statements, the counter-example is the case where "I do not work hard", "I do not earn lots of money", but "I pay high taxes".

□

(i) *If I like mathematics, then I will study.*
I will not study.
Either I like mathematics or I like football.
___
I like football.

Valid. Let p be the statement "I like mathematics", q be the statement "I will study", and r be the statement "I like football". Then the argument can be written as

$$\begin{array}{c} p \to q \\ \neg q \\ p \vee r \\ \hline r \end{array}$$

Assume that the assumptions are all true. Since the second assumption asserts that q is false, p must be false as well (cf. first assumption). Hence, r must be true (cf. third assumption). Since the conclusion r is true whenever the assumptions are all true, the argument is **valid**.

□

2. *Determine the truth value of* $[p \vee (q \to ((\neg r) \wedge s))] \leftrightarrow (r \wedge t)$*, where* p*,* q*,* r*,* s*, and* t *are all true.* Assume p, q, r, s, t are all true. Then the left hand side of the double implication is true (since p is true) and the right hand side is also true (since both r and t is true). Hence the double implication is true (since $LHS \to RHS$ is true and $RHS \to LHS$ is true).

□

3. *Determine if each statement is a tautology, a contradiction, or neither.*

(b) $(p \to q) \to (p \vee q)$. **Neither**. If p and q are both false then the statement is false, but if p is false and q is true then the statement is true, for example. Since the statement can take more than one truth value, it is neither a contradiction (*i.e.*, a statement that can only take F) nor a tautology (*i.e.*, a statement that can only take T).

□

(d) $[p \vee q] \leftrightarrow [(\neg q) \wedge r]$. **Neither.** If p, q, r are all true, then the statement is false (since $LHS \to RHS$ is false). If $p, q, r,$ are all false then the statement is true. Since the statement can take more than one truth value, it is neither a tautology nor a contradiction.

□

5a. *Suppose* $\mathcal{A}, \mathcal{B}, \mathcal{C}$ *are compound statements such that* $\mathcal{A} \iff \mathcal{B}$ *and* $\mathcal{B} \iff \mathcal{C}$*. Explain why* $\mathcal{A} \iff \mathcal{C}$ *(that is, give an elementary proof).* The first logical equivalence states that $\mathcal{A}$ takes identical truth values as $\mathcal{B}$, while the second states that $\mathcal{B}$ takes identical truth values as $\mathcal{C}$. Hence, the truth values of $\mathcal{A}$ and $\mathcal{C}$ are identical as well.

□

8a. *Determine if the argument is valid.* $\begin{array}{c} p \to q \\ \neg p \\ \hline \neg q \end{array}$. **Invalid.** If p is false and q is true, then the assumptions are both true but the conclusion is false.

3. *Prove that it is possible to fill an order of $n \geq 32$ pounds of fish given an unlimited supply of fish that are 5-pounds or 9-pounds.* We prove by induction. If $n = 32$ then we can fill the order for 32 pounds of fish using three 9-pound fish and one 5-pound fish since $3(9) + 1(5) = 32$. Now suppose that $n > 32$ and we know how to fill an order for $n - 1$ pounds of fish, and we need to show how to fill an order for n pounds of fish. Let x be the number of 9-pound fish and y be the number of 5-pound fish needed to fill the order for $n-1$ pounds of fish. If $x \geq 1$, then we can replace one 9-pound fish with two 5-pound fish to increase the total number of pounds by one, hence we can fill the order for n pounds of fish. Otherwise, $x = 0$ and the entire order for $n - 1$ pounds of fish is filled using only 5-pound fish. Hence, $n - 1 \geq 35$ since $32, 33, 34$ are not divisible by 5. So in this case, $y \geq 7$ and we can replace seven 5-pound fish with four 9-pound fish, and again increase the total number of pounds by one. Thus, we can always fill n pounds of fish if we can fill $n - 1$ pounds of fish for $n > 32$. This completes the proof by induction.

□

4d *Use mathematical induction to prove the statement for all $n \geq 1$: $8^n - 3^n$ is divisible by 5.* We prove by induction. If $n = 1$ then $8^1 - 3^1 = 5$ is divisible by 5, so the base case $n = 1$ is verified. Now assume that $8^{n-1} - 3^{n-1}$ is divisible by 5 and we need to show that $8^n - 3^n$ is divisible by 5. Since $8^{n-1} - 3^{n-1}$ is divisible by 5, there exists a natural number $k \geq 1$ such that $8^{n-1} - 3^{n-1} = 5k$. Now,

$$\begin{aligned}
8^n - 3^n &= 8 \cdot 8^{n-1} - 3 \cdot 3^{n-1} \\
&= (5+3) \cdot 8^{n-1} - 3 \cdot 3^{n-1} \\
&= 5 \cdot 8^{n-1} + 3 \cdot 8^{n-1} - 3 \cdot 3^{n-1} \\
&= 5 \cdot 8^{n-1} + 3(8^{n-1} - 3^{n-1}) \\
&= 5 \cdot 8^{n-1} + 3 \cdot 5k \\
&= 5(8^{n-1} + 3k).
\end{aligned}$$

So, $8^n - 3^n$ is divisible by 5 (since it is 5 times a natural number, the fact that $8^{n-1} + 3k$ is a natural number is left as an exercise to the reader).

□

5a *Prove by mathematical induction that for any natural number n we have*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

I'd like to remark that there is an elementary proof of this fact, the validity of which was known already to Pythagorus in the 6th century B.C.E. The following proof is attributed to Gauss, but it is essentially folklore at this point and it goes like this "Notice that there are $n/2$ pairs of terms which sum to $n+1$. Q.E.D." As the story goes, Gauss gave the proof to his 3rd grade teacher. Anyway, the proof of the statement using mathematical induction follows.

If $n = 1$ then the statement is true (also ok to use $n = 0$ instead as the base case, it just depends on whether you include zero as a natural number or not). Now suppose that $n > 1$ and

$$1 + 2 + \cdots + (n-1) = \frac{(n-1)((n-1)+1)}{2}.$$

Let's add n to both sides of the equality. We obtain

$$1 + 2 + \cdots + (n-1) + n = \frac{(n-1)n}{2} + n,$$

and simplifying the right hand side of the above by finding a common denominator, we see that

$$\frac{(n-1)n}{2} + n = \frac{(n-1)n}{2} + \frac{2n}{2} = \frac{(n-1)n + 2n}{2} = \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2},$$

as desired.

□

13 *What is wrong with the following "proof" (see book for the paragraph which you are supposed to analyze).* In the given proof, the author states that "we can fill an order for $k-5$ pounds of fish" for any $k > 10$, but this assertion is not true if k is in the range $11 \leq k \leq 14$, for example. In particular, the author would have needed to verify that orders of size 6 to 9 could be filled, when it is clear that they can not. Said another way, the crucial point is that the induction hypothesis can not be applied to $k = 14$, for example, since the base case was not checked for an order of size $k - 5 = 9$.

□