Homework 5: Core solutions

Section 6.1 on page 190-192 problems 1c, 1d, 3a, 3d, 10b.
Section 6.2 on page 198-199 problems 4a, 4b, 6a, 8a-8d, 12, 17a-17d.
Section 6.3 on page 202-203 problems 2, 6(do the $n = 5$ case first), 8a, 8b.
Review exercises for Chapter 6 problem 3b.

1. *There are 15 people who like pizza: ten like Canadian bacon, seven like anchovies, and six like both.*

(c) *How many like exactly one topping? Solution:* Let C be the set of people who like Canadian bacon, A the set who like anchovies. We have from the problem that $|C| = 10$, $|A| = 7$ and $|C \cap A| = 6$. So, there is 1 person who likes anchovies but not Canadian bacon, and 4 people who like Canadian bacon but not anchovies. In total, there are 5 people who like only one of the toppings.

$\square$

(d) *How many like neither topping? Solution:* From the inclusion-exclusion principle there are 11 people in the set $C \cup A$, who like one or the other. Therefore, there are $15 - 11 = 4$ people who like neither topping.

3. *Among a group of 30 students 15 people know JAVA, 12 know HTML, and 5 know both languages.*

(a) *How many students know at least one language? Solution:* Let J be the set of students who know JAVA and H be the set who know HTML. Then by the inclusion-exclusion principle $|J \cup H| = |J| + |H| - |J \cap H| = 15 + 12 - 5 = 22$. So 22 people know at least one language.

(d) *How many students know exactly one language?* One way to get this is to take 22 and subtract the number of people that know both (again). You get $22 - 5 = 17$.

10b. *Suppose U is a collection of objects (a set) containing 75 elements and A_1, A_2, A_3, and A_4 are sub-collections (subsets) of the elements of U with the following properties.*

- *Each subset contains exactly 28 elements.*
- *The intersection of any two subsets contains exactly 12 elements.*
- *The intersection of any three of the subsets contains exactly 5 elements.*
- *The intersection of all four subsets contains exactly 1 element.*

How many elements are in exactly two of the subsets A_1, A_2, A_3, A_4***?*** *Solution:* Let's first try to find the number elements that are in A_1 and A_2 but not in A_3 nor A_4. That is, let's count how many are in $A_1 \cap A_2$ but not in $A_3 \cup A_4$.

One way to do this is to think about the "disjunctive normal form using long minterms" of $A_1 \cap A_2$. That is, the set $A_1 \cap A_2$ can be written as

$$A_1 \cap A_2 = (A_1 \cap A_2 \cap A_3 \cap A_4) \cup (A_1 \cap A_2 \cap A_3^c \cap A_4) \cup (A_1 \cap A_2 \cap A_3 \cap A_4^c) \cup (A_1 \cap A_2 \cap A_3^c \cap A_4^c),$$

where A_i^c denotes the *complement* of the set A_i, that is, the elements which are *not* in A_i.

Note that each of the four sets in the union above is *disjoint* from the others, meaning that they have no elements in common. Thus, the number of elements in $A_1 \cap A_2$ (which was 15 by assumption) is equal to summing up four terms, one for each set appearing in parentheses in the union above (that is, you don't have to worry about counting the various intersections because there is none when the sets are disjoint).

We have

$$\begin{aligned} |A_1 \cap A_2| &= 12, \\ |A_1 \cap A_2 \cap A_3| &= 5, \\ |A_1 \cap A_2 \cap A_4| &= 5, \\ |A_1 \cap A_2 \cap A_3 \cap A_4| &= 1. \end{aligned}$$

So, it is easy to see that $|A_1 \cap A_2 \cap A_3^c \cap A_4| = 4$, and also $|A_1 \cap A_2 \cap A_3 \cap A_4^c| = 4$. Indeed, out of the 5 elements in $A_1 \cap A_2 \cap A_3$, exactly one of them is also in A_4. Finally, we have

$$\begin{aligned} |A_1 \cap A_2| &= |A_1 \cap A_2 \cap A_3 \cap A_4| \cup |A_1 \cap A_2 \cap A_3^c \cap A_4| \cup |A_1 \cap A_2 \cap A_3 \cap A_4^c| \cup |A_1 \cap A_2 \cap A_3^c \cap A_4^c|, \\ 15 &= 1 + 4 + 4 + |A_1 \cap A_2 \cap A_3^c \cap A_4^c|. \end{aligned}$$

So $|A_1 \cap A_2 \cap A_3^c \cap A_4^c| = 3$.

Now just note that $A_1 \cap A_2 \cap A_3^c \cap A_4^c$ is exactly the set of elements in $A_1 \cap A_2$ that are not in $A_3 \cup A_4$. So the number we were looking for is 3.

Now, let's finish the problem.

It stands to reason that the analysis we just performed had nothing to do with the choice of A_1, A_2 as the sets we were including and A_3, A_4 as the sets we were not including (if you want to be fancy when you are making this argument, just say "Without loss of generality the same is true for any other choice ...", or if you *really* want to be fancy just write WLOG). So, any other choice also yields 6 elements. There are $\binom{4}{2} = 6$ combinations, so the total number of elements in exactly two sets is $3 \cdot 6 = 18$. $\square$

4a. *How many numbers in the range 100-999 have no repeated digits? Solution:* Starting with the hundreds place, there are 9 choices for a digit. For the tens place, there are again 9 choices since we only can not pick the digit in the hundreds place. In the ones place, there are 8 choices since we can not pick either of the digits in the hundreds nor tens place and those digits must be distinct by construction. So by the multiplication rule there are $9 * 9 * 8 = 648$ numbers in this range with no repeated digits.

□

4b. *How many* ***odd*** *numbers in the range 100-999 have no repeated digits?. Solution:* It seems it should be half of 648, but this doesn't work.

There are 5 choices for the ones digit, then 8 choices for the hundreds digit, and finally 8 choices for the tens digit. So by the multiplication rule there are $8 * 8 * 5 = 320$ odd numbers in this range with no repeated digits.

Remark: If you pick digits in the order: ones digit, tens digit, hundreds digit, then the count becomes $5 * 8 + 5 * 8 * 7$ since there are two cases: either you pick a 0 in the tens digit or not, respectively (per summand).

6a. *How many five-letter palindromes (not necessarily real words) can be made from the letters of the English language? Solution:* A *palindrome* is spelled the same frontwards and backwards. Thus, we need only choose which letter is first, second and third (the fourth and fifth letter are determined by the second and first letters, respectively). There are 26 choices for each of the first, second, and third letter, so there are $26^3 = 17576$ five-letter palindromes.

□

8. *From a standard deck of 52 playing cards, find how many ways are there to do each of the following.*

(a) *Draw a heart or a spade. Solution:* There are 13 hearts and 13 spades, and no card is both, so there are 26 ways to do this.

(b) *Draw an ace or a king. Solution:* There are 4 aces and 4 kings, and no card is both, so there are 8 ways to do this.

(c) *Draw a card numbered 2 through 10.* There are 9 cards in that range in each of four suits, so there are 36 ways to do this.

(d) *Draw a card numbered 2 through 10 or a king.* There are 10 cards in that range in each of 4 suits, so there are 40 ways to do this.

12. *In how many ways can two adjacent squares be selected from an 8×8 chessboard? Solution:* We prove that for an $n \times n$ chessboard, $n \geq 2$, there are $2n(n-1)$ ways to select two adjacent squares (by the way, I got this number *as well as the idea for the proof* after looking at the cases $n = 2$ and $n = 3$). The proof is by induction.

For $n = 2$ there are 4 ways to pick adjacent squares (two horizontal pairs and two vertical pairs). Now suppose the number of ways to pick adjacent squares from a $k \times k$ chessboard is $2k(k-1)$. Consider a $(k+1) \times (k+1)$ sized chessboard. Thinking of the lower left-hand side as a $k \times k$ chessboard, there are $2k(k-1)$ pairs coming only from this $k \times k$ square (not including the top and right borders), by the induction hypothesis. There are an additional $2k$ pairs coming from the top border, and an additional $2k$ pairs coming from the right border (draw a picture if you don't see this yet). So there are $2k(k-1) + 4k = 2(k+1)k$ pairs total for the $(k+1) \times (k+1)$ sized board, which is what we needed to show.

$\square$

17. *Two fair dice are rolled.*

(a) *In how many ways can a total of 8 arise? Solution:* The two die can show 6 and 2 two ways, 5 and 3 two ways, and 4 and 4 one way. So there are 5 ways to roll an 8.

(b) *In how many ways can a total of 7 arise? Solution:* There are 6 choices for the first die, and then the second die is determined if we need a to roll a 7. So there are 6 ways to roll a 7.

(c) *In how many ways can a total of 7 or 8 arise? Solution:* There are $5 + 6 = 11$ ways to roll a 7 or 8 (this events are mutually exclusive *or disjoint if you like that term better*).

(d) *In how many ways can doubles arise? (Doubles means that both die show the same number of pips) Solution:* There are 6 ways to get doubles. Once you know which of the six numbers is on the first die, the number of pips on the second die is determined.

2. *Prove that in any string of six integers there is a substring, possibly consisting of just one number, whose sum is divisible by 6. Solution:*

As an aside: I had to think for a minute about this one. Then I realized something that was super useful. Before going into the solution, let me try to explain what I saw. I wrote down the following example, as the book indicates is a good idea to try: my six integers that I picked were

$$7, 13, 21, 29, 33, 35.$$

I picked ones that were not divisible by sex, else it would be too easy. I tried adding up a few adjacent pairs, but didn't see any that gave me something divisible by 6. Then I added them all up and, to my amazement, got 138 which is $6 * 23$. Ok, then I took a walk and realized that I should be using *remainders*. The sequence of numbers I picked has the following remainders, located below the number.

Number	7	13	21	29	33	35
Remainder	1	1	3	5	3	5

Now it is easy to see what is happening. When I added all the numbers to get 138 I was adding together a bunch of numbers which were divisible by 6 and a bunch of remainders that summed to $1 + 1 + 3 + 5 + 3 + 5 = 18$, which is also divisible by 6. It was much easier for me to see that no adjacent numbers would work, since in this case the sum of the remainders of adjacent numbers are $1 + 1 = 2, 1 + 3 = 4$, and $3 + 5 = 8$, none of which are divisible by 6. I tried a couple things trying to take advantage of this new perspective, but nothing worked. Finally, I checked the book for an idea and then saw the example of Problem 13 on page 200. The following is the argument from the example in the book, modified to this problem.

Let the six numbers in the list be denoted by $n_1, n_2, n_3, n_4, n_5, n_6$. Consider the 6 numbers

$$\begin{aligned}
&(1) \quad n_1,\\
&(2) \quad n_1 + n_2,\\
&(3) \quad n_1 + n_2 + n_3,\\
&(4) \quad n_1 + n_2 + n_3 + n_4,\\
&(5) \quad n_1 + n_2 + n_3 + n_4 + n_5,\\
&(6) \quad n_1 + n_2 + n_3 + n_4 + n_5 + n_6.
\end{aligned}$$

Each of these six numbers has some remainder after dividing by 6 (also known as modulo class or *congruence class*). The remainders are in the range 0,1,. . . ,6, but if any of them are 0 then we are done since the conclusion we are trying to reach is satisfied by the sum above corresponding to this remainder. If none of them are 6, then by the pigeon hole principle there must be at least two repeated remainders. If numbers $(i), (j)$ above with $i > j$ have the same remainder, then the number $(i) - (j)$ has the desired property: namely

$$(n_1 + n_2 + \ldots + n_i) - (n_1 + n_2 + \ldots + n_j) = n_{j+1} + \ldots + n_i$$

is divisible by 6.

To see this last part: I will show that if two integers a, b have the same remainder after dividing by an integer n, then the difference $a - b$ is divisible by n. Indeed, write $a = nd_1 + r$ and $b = nd_2 + r$ for integers d_1, d_2, r with the remainder r in the range $0 \leq r \leq n$. Then

$$a - b = (nd_1 + r) - (nd_2 + r) = n(d_1 - d_2) + (r_1 - r_2) = n(d_1 - d_2).$$

$\square$

6. *In a standard 52 card deck of playing cards, how many cards of a single suit must be present in any set of six cards? How many cards of a single suit must be present in any set of n cards? How many cards of a single denomination must be present in a set of six cards? How many cards of a single denomination must be present in a set of n cards? Solution:* There must be at least one suit with two or more cards in any hand of six cards, using the pigeon hole principle. For a hand of n cards, there must be at least one suit with $\lceil \frac{n}{4} \rceil$ or more cards of that suit (so in particular when $n = 6$ the answer is $\lceil \frac{6}{4} \rceil = \lceil 1.5 \rceil = 2$.

Ditto for denominations instead of suits. For n cards it is $\lceil \frac{n}{13} \rceil$, so when $n = 6$ there is at least one denomination that has at least one card of that denomination (which is not really saying very much, really). $\square$

8. *Thirty buses are to be used to transport 2000 refugees. Each bus has 80 seats. Assume there is only 1 seat per passenger.*

(a) *Prove that one of the buses will carry at least 67 passengers. Solution:* By the strong form of the pigeon hole principle, there must be at least $\lceil \frac{2000}{30} \rceil = \lceil 66.667 \rceil = 67$ passengers on at least one bus.

(b) *Prove that one of the buses will have at least 14 empty seats. Solution:* Suppose not. Then all the busses have 13 or less empty seats, and in this case there are at least $30 * (80 - 13) = 30 * 67 = 2010$ passengers, which is a contradiction.

3b. *How many seven-letter palindromes (not necessarily real English words) begin with the letter S and contain at most three different letters? Solution:* The first and last letter must be S, and we can choose at most two more letters. If we choose zero more letters (all letters are

S) then the word is SSSSSSS. If we chose one more letter, then the word is either SxSSSxS, SSxSxSS, SSSxSSS, SxxSxxS, SxSxSxS, SSxxxSS, or S$xxxxx$S, and each of these 7 cases has 25 possible palindromes depending on what letter x is (remember it can't be S). If we chose two more letters instead of one, then the word is either SxySyxS, SxSySxS, SSxyxSS, S$xyyyx$S, S$xxyxx$S, or S$xyxyx$S, and each of these 6 cases has $25 * 24 = 600$ different palindromes possible. In all, there are $1 + 7 * 25 + 6 * 600 = 3776$ different seven-letter palindromes that begin with S and have at most three different letters.

$\square$