Homework 6: Core solutions

Section 7.1 on page 209-210 problems 2, 4, 6, 7a, 7b, 11
Section 7.2 on page 215-216 problems 2a, 2b, 4, 9, 11.

2. *Eight horses enter a race in which first, second, and third prizes are awarded. Assuming no ties, how many outcomes are possible?* There are $\binom{8}{3}$ ways to pick three winners out of eight horses, and 3! ways to arrange the three winners into first, second and third. Thus, by the multiplication principle there are

$$3!\binom{8}{3} = 3!\frac{8!}{5!3!} = \frac{8!}{5!} = 8 * 7 * 6 = 336$$

outcomes possible. *Alternate solution:* There are 8 ways to pick the first place horse, then 7 ways and 6 ways to pick the second and third place horses, respectively. Thus, by the multiplication rule there are $8 * 7 * 6 = 336$ possible outcomes.

□

4. *Eight students leave a review session in the first 15 minutes. If there were 30 students at the beginning of the session, in how many ways could this exodus occur?* There are $\binom{30}{8} = 5852925$ ways for this to happen.

6. *Four cats and five mice enter a race. In how many ways can they finish with a mouse placing first, second, and third?* There are $3!\binom{5}{3}$ ways to pick which mouse comes in first, second, and third, and there are $(4+2)! = 6!$ ways to arrange the 4 cat and 2 mouse "losers". Hence, by the multiplication rule there are

$$3!\binom{5}{3} \cdot 6! = 5 * 4 \cdot 6 * 5 * 4 * 3 = 7200$$

ways for the race to end.

□

7a. *In how many ways can 10 boys and 4 girls sit in a row?* There are $(10+4)! = 14!$ ways for 14 people to sit in a row.

7b. *In how many ways can 10 boys and 4 girls sit in a row if all the boys must sit together and so must all the girls?* There are 10! ways for 10 boys to sit in a row, 4! ways for 4 girls to sit in a row, and 2 ways for either boys on the left or boys on the right. By the multiplication rule there are $2 \cdot 10!4!$ ways for them to sit if the boys and girls must each sit together. □

11. *How many permutations of the letters a, b, c, d, e, f, g contain neither the string bge nor the string eaf?* There are 7! permutations of the seven letters. Among these, lets count how many contain the string bge. There are 4! ways to permute the letters a, c, d, f (those that are not b, g, e), and for each of these 4! permutations there are 5 places that you could insert the string bge. Hence, there are $5 \cdot 4! = 5!$ different strings that contain bge. Similarly, there are 5! different strings that contain eaf since there was nothing special about the particular choice of b, g, e in our argument. Now, in order for the string of seven letters to contain BOTH the string bge AND the string eaf, since e is a common letter to both strings the seven letter string must contain the 5 letter string $bgeaf$. There are only 6 ways for this to happen (either the seven letter string is $\boldsymbol{cd}bgeaf$, $\boldsymbol{dc}bgeaf$, $\boldsymbol{c}bgeaf\boldsymbol{d}$, $\boldsymbol{d}bgeaf\boldsymbol{c}$, $bgeaf\boldsymbol{cd}$, or $bgeaf\boldsymbol{dc}$). Thus, by the inclusion-exclusion principle, the number of strings that contain either the one string bge, the other string eaf, or both $bgeaf$ is

$$5! + 5! - 6 = 234.$$

Thus, the number of seven letter strings that contain neither bge nor eaf is $7! - 234 = 4806$.

□

2a. *How many ways can 12 people be split evenly into two groups, the red group and the blue group?* There are $\binom{12}{6}$ ways to select 6 people for the red group. The other people go in the blue group. So the answer is $\binom{12}{6} = 924$.

2b. *How many ways can 12 people be split evenly into two groups?* This blew my mind. There are 462 ways to split up 12 people into two groups of size 6.

Remark: Let's do an illustrative example. Suppose there are 4 pieces of fruit in the fruit bowl at my house: an apple a, a banana b, a clementine c, and a donut d (ok, I know a donut is not a fruit, but the only other fruit I know that starts with d is durian). I want to take 2 pieces of fruit for lunch. How many ways are there to do this? I can take any of the following:

$$ab, ac, ad, bc, bd, \text{ or } cd.$$

So there are 6 ways to take two fruit. But how many ways are there to make two groups of 2 fruit each? There are only 3 ways! Either the apple is grouped with the banana, the clementine, or the donut! So what happened? When I additionally have to chose "my" group I can count by thinking "in each of the 3 ways there are to make two groups of 2 pieces of fruit, which of the two groups is mine". Anyway, back to the solution.

There are 462 ways to split up 12 people into two groups of size 6. There are $\binom{12}{6}$ ways to split up people into a first and second group, but there were 2 ways to label which was first and which was second, so there are $\binom{12}{6}/2 = 462$ ways to evenly split up a group of 12 people.

□

4. *How many signals, each consisting of seven flags arranged in a row, can be formed from three identical red flags and four identical blue flags?* There are 7! ways to arrange seven objects. Suppose we arrange the seven flags to get a particular signal. Then, there are 4! ways to rearrange the blue flags so that we have the same signal (switching 2 blue flags does not change the signal). Similarly, there are 3! ways to rearrange the red flags without changing the signal. So there are 4!3! ways to rearrange the flags without changing the signal. So the total number of signals possible is

$$\frac{7!}{4!3!} = \binom{7}{3} = \binom{7}{4} = 35,$$

corresponding to choosing the 3 locations of the red flags (or the 4 locations of the blue flags).

□

9. *A club of 30 people wants to elect a board consisting of a president, secretary, treasurer, and two undersecretaries. In how many ways can the board be selected?* There are 30 ways to choose the president, then 29 ways to chose the secretary once a president is chosen, then 28 ways to chose a treasurer once the president and secretary have been chosen, and finally $\binom{27}{2} = 27 * 26/2 = 351$ ways to chose the two undersecretaries out of the remaining 27 non-elected members. Thus, the total number of ways for a board to be chosen is $30 * 29 * 28 * 27 * 26/2 = 8550360$.

□

11. *An urn contains 15 red numbered balls and 10 white numbered balls. Five balls are sampled, removed from the urn and placed side-by-side on a table. How many different samples are possible? How many samples contain only red balls? How many samples contain exactly three red balls and two white balls?* There are $\binom{25}{5}$ samples possible. There are only $\binom{15}{5}$ of these samples that contain only red balls. There are $\binom{15}{3}\binom{10}{2}/(3!2!)$ samples with exactly three red balls and two white balls.

Remark: About the factor of $^1/_{5!}$ in the last part. Think about the question: *How many ways are there to pick out the 5 balls in your sample?* If you thought that the answer to the last part of problem **11.** was $\binom{15}{3}\binom{10}{2}$, then you were over counting by a factor of 5!. Again, a illustrative example should clear things up a bit. Suppose the urn has only two balls: one red and one white. You pick two balls out of the urn, the sample you have has one red ball and one white ball, but there were two ways to get it out of the urn, either you picked the red ball first or the white ball first.