Homework 7: Core solutions

Section 5.2 on page 167 problems 1b-1d, 3, 12a-12c, 20a-20b.
Section 5.3 on page 174 problems 2, 8, 13.
Section 5.4 on page 181 problems 1a-1c.

1. *Give recursive definitions of the following sequences.*

(b) 5, 3, 1, -1, -3, ... *Solution:* We have that the sequence is defined recursively as $a_n = a_{n-1} - 2$ and $a_1 = 5$. Solving the recurrence we see that $a_n = 5 - 2(n-1)$.

(c) 4, 1, 3, -2, 5, -7, 12, -19, 31, ... *Solution:* The sequence is defined recursively by $a_n = a_{n-2} - a_{n-1}$, for all $n \geq 3$ and $a_1 = 4$, $a_2 = 1$.

(d) 1, 2, 0, 3, -1, 4, -2, ... *Solution:* This one was definitely the hardest of the three to figure out. The reason is because the depth of the recurrence is 3 and those are more difficult to see. The sequence is recursively defined by $a_n = a_{n-3} + a_{n-2} - a_{n-1}$ for $n \geq 4$, with $a_1 = 1$, $n_2 = 2$, $n_3 = 0$.

□

3. *Define a sequence recursively by $a_1 = 1$ and $a_n = 3a_{n-1}$ for $n \geq 2$. Solve the recurrence relation. That is, find an explicit formula for a_n that does not involve any of the previous terms. Solution:* The solution is $a_n = 3^{n-1}$. This is proved by induction and the proof is left to the reader.

12. *Define a sequence recursively by $a_0 = 2$, $a_1 = 3$, and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$.*

(a) *Find the first 5 terms. Solution:* The first 5 terms are 2, 3, 5, 9, 17.

(b) *Guess a formula for a_n. Solution:* Looking at the progression in the first 5 terms, we see that to get from the first term to the second you add 1, to get from the second to the third you add 2, to get from the third to the fourth you add 4, and to get from the fourth to the fifth you add 8. But we are asked to guess a *solution*, not a recursive definition. With this perspective, and using what we already discovered, we guess that $a_n = 1 + 2^n$ for $n \geq 0$.

(c) *Prove that your guess is correct. Solution:* We just check that the definition $a_n = 2 + 2^n$ satisfies the recurrence relation. The base cases are when $n = 0$ and $n = 1$, and we have that $a_0 = 1 + 2^0 = 2$, which is the correct value for the 0-th term of the sequence provided, and $a_1 = 1 + 2^1 = 3$, which is also the correct value. Now, given $a_n = 1 + 2^n$ we have to check that $a_n = 3a_{n-1} - 2a_{n-2}$. We have $3a_{n-1} - 2a_{n-2} = 3(1+2^{n-1}) - 2(1+2^{n-2}) = 1 + 3 \cdot 2^{n-1} - 2 \cdot 2^{n-2} = 1 + 2 \cdot 2^{n-1} = 1 + 2^n$, as desired.

□

20. An arithmetic sequence begins 116, 109, 102,....

(a) *Find the 300th term. Solution:* An arithmetic sequence is one that is defined recursively as $a_n = a_{n-1} + c$ for some constant c. In this case, $c = -7$ and the solution to the recursion is $a_n = 116 - 7(n-1)$, for $n \geq 1$. Thus, the 300th term is $a_{300} = 116 - 7(300-1) = 116 - 7(299) = -1977$.

(b) *Is -480 a term? If yes, which one. Solution:* Note that -480 has a remainder of 3 modulo 7 (since $-480 = 7 * (-68) + 3$). But all the terms in the sequence have a remainder of 4 modulo 7 (for example, $116 = 7 * 16 + 4$). So, -480 is not a term of the sequence. The closest terms of the sequence to -480 are -476 and -483.

□

2. *Solve the recursively defined sequence $a_n = -6a_{n-1} + 7a_{n-2}$, for $n \geq 2$ with $a_0 = -5$ and $a_1 = 3$. Solution:* The characteristic polynomial is $x^2 + 6x - 7$ which factors as $(x-1)(x+7)$, so the roots of the characteristic polynomial are $x_1 = 1$ and $x_2 = -7$. Hence, the general solution to the recursion is $a_n = c_1(1)^n + c_2(-7)^n$. Next we use the known values of a_0, a_1 to solve for c_1, c_2. We have

$$a_0 = -5 \implies c_1 + c_2 = 5, \text{ and}$$
$$a_1 = 3 \implies c_1 - 7c_2 = 3.$$

We have to solve two linear equations in two variables, which can be done using several methods. For example, adding 7 times the first equation to the second equation we see that $8c_1 = 38$ or $c_1 = 19/4$, and then substituting this into the first equation we get $c_2 = 1/4$. So the particular solution to the recurrence is $a_n = \frac{19}{4} + \frac{1}{4}(-7)^n$.

□

8. *Solve the recurrence $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, where $a_0 = 40$ and $a_1 = 37$. Solution:* The characteristic polynomial is $x^2 - 2x + 1 = (x-1)^2$, so the general solution is $a_n = c_1 - nc_2$. The particular solution is found using the given values of a_0, a_1. We have

$$a_0 = 40 \implies c_1 - 0c_2 = 40, \text{ and}$$
$$a_1 = 37 \implies c_1 - c_2 = 37.$$

Therefore, $c_1 = 40$ and $c_2 = 3$, and the particular solution is $a_n = 40 - 3n$.

□

13a. *Solve the homogeneous recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, and $a_0 = 2$, $a_1 = 11$.* *Solution:* The recurrence is solved in the usual way. The characteristic polynomial is $x^2 - 5x + 6 = (x-6)(x+1)$ which has roots $x = 6, -1$. The general solution is thus $a_n = c_1 6^n + (-1)^n c_2$. We find c_1, c_2 using the given information that $a_0 = 2, a_1 = 11$. We have

$$a_0 = 2 \implies c_1 + c_2 = 2, \text{ and}$$
$$a_1 = 11 \implies 6c_1 - c_2 = 11.$$

Adding the two equations together we get $7c_1 = 13$, hence $c_1 = 13/7$. Substitution into the first equation yields $c_2 = 2 - 13/7 = 1/7$. So the particular solution is $a_n = \frac{13}{7}6^n + \frac{1}{7}(-1)^n$.

□

1. *What sequence is associated to the following generating functions?*

(a) $(2-3x)^2$ *Solution:* Expand the polynomial to see that it is $4 - 12x + 9x^2$, so the corresponding sequence with this generating function is 4, -12, 9, 0, 0, 0, ….

(b) $\frac{x^4}{1-x}$ *Solution:* Recall that $\frac{1}{1-x} = 1 + x + x^2 + \cdots$. So, $\frac{x^4}{1-x} = x^4(\frac{1}{1-x}) = x^4(1 + x + x^2 + \cdots) = x^4 + x^5 + x^6 + \cdots$. The corresponding sequence to this generating function is 0, 0, 0, 0, 1, 1, 1, ….

(c) $\frac{1}{(1+3x)^2}$ *Solution:* Note that $\frac{1}{1+3x} = \frac{1}{1-(-3x)} = 1 - 3x + (3x)^2 - (3x)^3 + \cdots$. Hence,

$$\begin{aligned}
\frac{1}{(1+3x)^2} &= \left(\frac{1}{1+3x}\right)\left(\frac{1}{1+3x}\right) \\
&= (1 - 3x + 9x^2 - 27x^3 + \cdots)(1 - 3x + 9x^2 - 27x^3 + \cdots) \\
&= 1 + (-3-3)x + (9+9+9)x^2 + \cdots \\
&= 1 - 6x + 27x^2 - 108x^3 + \cdots + (n+1)(-3)^n x^n + \cdots .
\end{aligned}$$

So, the corresponding sequence to this generating function is 1, -6, 27, -108, …, defined by $a_n = (n+1)(-3)^n$, $n \geq 0$.

□