Homework 9: Core solutions

Section 8.2 on page 264 problems 13b, 27a-27b.
Section 8.3 on page 275 problems 1b, 8, 10a-10b, 14.
Section 8.4 on page 279 problems 2c, 7a.
Chapter 8 review problems on page 280 problem 6.

13b. *Show that $b^n \prec n!$ for any $b > 1$ Solution:* Let $a_n = \frac{b^n}{n!}$. Then $\frac{a_{n+1}}{a_n} = \frac{b^{n+1}/(n+1)!}{b^n/n!} = \frac{b}{n+1}$, which converges to zero as n tends to infinity. By the ratio test (from Calculus), the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. A necessary condition for the series to converge, however, is that the terms a_n must tend to zero. Hence, $\lim_{n\to\infty} \frac{b^n}{n!} = 0$. By Proposition 8.2.6 on page 256, we have $b^n \prec n!$.

□

27a. *Establish the triangle inequality:*

$$|a + b| \leq |a| + |b|.$$

Solution: For any real number x and y we have

$$-|x| \leq x \leq |x|, \text{ and}$$
$$-|y| \leq y \leq |y|.$$

Adding the two inequalities together we have

$$-\left(|x| + |y|\right) \leq x + y \leq |x| + |y|. \tag{1}$$

But

$$|x+y| = \begin{cases} x+y & \text{if } x+y \geq 0 \\ -(x+y) & \text{if } x+y < 0. \end{cases}$$

In either case, $|x + y| \leq |x| + |y|$ using Equation 1.

Alternate proof: We first show that $|x + y|^2 \leq (|x| + |y|)^2$. We have

$$\begin{aligned} |x+y|^2 &= (x+y)(x+y) \\ &= x^2 + 2xy + y^2 \\ &= |x|^2 + 2xy + |y|^2 \\ &\leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2. \end{aligned}$$

Now, if a, b are any positive real numbers and $a \leq b$, then $\sqrt{a} \leq \sqrt{b}$. This is immediate since the function $f(x) = \sqrt{x}$ is strictly increasing on it's domain (recall, $f'(x) = \frac{1}{2\sqrt{x}} > 0$ for $x > 0$). This finishes the proof.

□

27b. *Show that* $|x_1+\cdots+x_n| \leq |x_1|+\cdots+|x_n|$ *for any* $n \geq 1$ *and real numbers* $x_1, \ldots, x_n$. *Solution:* Proof by induction. When $n = 1$ we have $|x_1| \leq |x_1$ ✓ (by the way, the $n = 2$ case is just what we did in the previous exercise). Now, suppose $|x_1 + \cdots + x_k| \leq |x_1| + \cdots + |x_k|$. We have,

$$\begin{aligned} |x_1 + \cdots + x_k + x_{k+1}| &\leq |x_1 + \cdots + x_k| + |x_{k+1}| \\ &\leq |x_1| + \cdots + |x_k| + |x_{k+1}, \end{aligned}$$

where the first inequality follows from the previous exercise and the second equality follows from the induction hypothesis.

□

1b. *Show the sequence of steps in a binary search to find* $x = 7$ *in the list* $1, 2, 3, 4, 5, 6, 7, 8, 9$. *How many times is* x *compared with an element in the list? How many times would it be compared if we used a linear search? Solution:* Recall the binary search and linear search algorithms.

Binary search algorithm:
Input: $a_1, \ldots, a_n, x$ with $a_1 \leq a_2 \leq \cdots \leq a_n$.
Procedure:
STEP 1: Initialize $S = 0$.
STEP 2: WHILE $n > 0$,
  IF $n = 1$ then
    IF $x = a_1$ set $n = 0$ and replace S with 1.
    ELSE set $n = 0$.
  ELSE
    set $m = \lfloor \frac{n}{2} \rfloor$;
    IF $x \leq a_m$ replace the current list with $a_1, \ldots, a_m$ and set $n = m$;
    ELSE replace the current list with $a_{m+1}, \ldots, a_n$ and replace n by $n - m$.
  END WHILE
Output: S.

Linear search algorithm:
Input: $a_1, \ldots, a_n, x$.
Procedure:
STEP 1: Initialize $S = 0$.
  For $i = 1..n$,

IF $x = a_i$, set $i = 2n$ and replace S with 1.

Output: S.

The algorithms above each take a list and output 1 if x is an element of the list and output 0 otherwise.

The steps in the binary search with the list $1, 2, 3, 4, 5, 6, 7, 8, 9$ and $x = 7$ is as follows: Initialize $S = 0$. Since $n \neq 1$ set $m = \lfloor \frac{9}{2} \rfloor = 4$. The number $x = 7$ does not satisfy $x \leq a_m$ since $7 \nleq a_4 = 4$. So we replace the list by $5, 6, 7, 8, 9$. Since $n \neq 1$ set $m = \lfloor \frac{5}{2} \rfloor = 2$. The number $x = 7$ does not satisfy $x \leq a_m$ since $a_2 = 6$, so we replace the list by $7, 8, 9$. Since $n \neq 1$ set $m = \lfloor \frac{3}{2} \rfloor = 1$. The number $x = 7$ does satisfy $x \leq a_1$ since $a_1 = 7$. Hence, we replace the list with the list "7". Now, $n = 1$ and $x = a_1$, so we replace S with 1. Finally, we output $S = 1$.

There are a total of 4 comparisons using the binary search.

If we were to do a linear search, there would be 7 comparisons (6 failed comparison and then a successful comparison).

□

8. *Show the sequence of steps involved in merging the sorted lists 2, 4, 4, 6, 8 and 1, 5, 7, 9, 10. How many comparisons are required? Solution:* Recall the merging algorithm.

Merging algorithm: $MERGE(\mathcal{L}_1, \mathcal{L}_2)$
Input: $\mathcal{L}_1 = (a_1, \ldots, a_s)$ and $\mathcal{L}_2 = (b_1, \ldots, b_t)$ with $a_1 \leq a_2 \leq \cdots \leq a_s$ and $b_1 \leq b_2 \leq \cdots \leq b_t$.
Procedure:
STEP 1: Initialize $\mathcal{L}_3 = ()$.
STEP 2:
IF $\mathcal{L}_1$ is empty, set $\mathcal{L}_3 = \mathcal{L}_2$ and STOP.
IF $\mathcal{L}_2$ is empty, set $\mathcal{L}_3 = \mathcal{L}_1$ and STOP.
STEP 3:
IF $a_1 \leq b_1$, remove a_1 from $\mathcal{L}_1$ and append it to $\mathcal{L}_3$; if this empties $\mathcal{L}_1$ then append $\mathcal{L}_2$ to $\mathcal{L}_3$ and STOP. Relabel the elements in $\mathcal{L}_1$ and repeat Step 3.
ELSE $a_1 > b_1$, remove b_1 from $\mathcal{L}_2$ and append it to $\mathcal{L}_3$; if this empties $\mathcal{L}_2$ then append $\mathcal{L}_1$ to $\mathcal{L}_3$ and STOP. Relabel the elements in $\mathcal{L}_2$ and repeat Step 3.
Output: $\mathcal{L}_3$.

The sequence of steps is as follows:

	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$
1	$(2,4,4,6,8)$	$(1,5,7,9,10)$	$()$
2	$(2,4,4,6,8)$	$(5,7,9,10)$	(1)
3	$(4,4,6,8)$	$(5,7,9,10)$	$(1,2)$
4	$(4,6,8)$	$(5,7,9,10)$	$(1,2,4)$
5	$(6,8)$	$(5,7,9,10)$	$(1,2,4,4)$
6	$(6,8)$	$(7,9,10)$	$(1,2,4,4,5)$
7	(8)	$(7,9,10)$	$(1,2,4,4,5,6)$
8	(8)	$(9,10)$	$(1,2,4,4,5,6,7)$
9	$()$	$(9,10)$	$(1,2,4,4,5,6,7,8)$
10	$()$	$()$	$(1,2,4,4,5,6,7,8,9,10)$

There are 8 comparisons required (the 1rst and 10th step did not require a comparison).

□

10. *Find an example of two ordered lists of lengths s and $t \geq 3$ that can be merged with*

(a) *one comparison. Solution:* Set $\mathcal{L}_1 = (1)$ and $\mathcal{L}_2 = (2,3,4)$.

(b) *t comparisons. Solution:* Set $\mathcal{L}_\infty = (t+1)$ and $\mathcal{L}_2 = (1,2,3,\dots,t)$.

11. *Sort the list $7,2,2,5,3,5,4$ using bubble sort and merge sort. In each case, how many comparisons were needed? (for merge sort, you may ignore comparisons required to check the size and parity of n at each iteration of Step 3) Solution:* The merge sort and bubble sort algorithms are stated below. Recall the *Merging algorithm* above whose input $\mathcal{L}_1, \mathcal{L}_2$ are two ordered lists and whose output $MERGE(\mathcal{L}_1, \mathcal{L}_2)$ is the merged ordered list coming from $\mathcal{L}_1, \mathcal{L}_2$.

Merge sort algorithm:
Input: An unordered list $\mathcal{L} = (a_1, \dots, a_n)$.
Procedure:
STEP 1: Initialize $F = 0$.
STEP 2: For $i = 1..n$,
define $\mathcal{L}_i$ to be the list with single element a_i.
STEP 3: WHILE $F = 0$,
IF $n = 1$, set $F = 1$.
IF $n = 2m$ is even,

For $i = 1..m$

replace $\mathcal{L}_i$ with $MERGE(\mathcal{L}_{2i}, \mathcal{L}_{2i-1})$.

Set $n := m$.

IF $n = 2m + 1$ is odd and $n \neq 1$,

For $i = 1..m$

replace $\mathcal{L}_i$ with $MERGE(\mathcal{L}_{2i}, \mathcal{L}_{2i-1})$.

Set $\mathcal{L}_{m+1} := \mathcal{L}_n$.

Set $n := m + 1$.

Output: $\mathcal{L}_1$.

Bubble sort algorithm:
Input: An unordered list $\mathcal{L} = (a_1, \ldots, a_n)$.
Procedure:
STEP 1:
For $i = n - 1$ down to 1,

For $j = 1..i$,

IF $a_j > a_{j+1}$, set $a_j := a_{j+1}$ and $a_{j+1} := a_j$.

Output: $\mathcal{L}$.