Practice Exam 1

1. *Exploring: probability, distribution, random variable, expectation.* Consider the space which consists of the first 6 letters of the alphabet $S = \{a, b, c, d, e, f\}$. What does it mean to say that P is a probability on S?

Solution: P is a probability on S if it is a function which assigns to each event $E \subseteq S$ a number $P(E) \in \mathbb{R}$ such that (note that we need S to be finite for the way we have stated the third property to be enough)

- $P(E) \geq 0$,
- $P(S) = 1$,
- if $A_1, A_2, \ldots, A_k$ are events and $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = \sum_{i=1}^{k} P(A_i).$$

(i) Let P_1 be the probability giving the uniform distribution on S. Describe P_1. What is $P_1(a)$? What is $P_1(\text{vowel})$?

Solution: The probability which is evenly distributed across each element of the space S assigns a probability of $1/6$ to each of the six events. So, $P_1(a) = 1/6$ and $P_1(vowel) = P(\{a, e\}) = 1/3$.

(ii) Let P_2 be the probability for which each and any vowel has twice the probability of each and any consonant. (*Hint:* $P_2(a) = 2P_2(b)$, *for example*) What is $P_2(S)$? $P(b)$? What is $P_2(\text{vowels})$? What is $P_2(\{a, f\})$? $P_2(\{a, b, c\})$?

Solution: Like every probability, $P_2(S) = 1$. Note that $P(a) = P(e) = 1/4$ and $P(b) = P(c) = P(d) = P(f) = 1/8$. Now $P_2(\text{vowels}) = P_2(\{a, e\}) = 1/2$ and $P(\{a, f\}) = 3/8$ and $P(\{a, b, c\}) = 1/2$.

(iii) Let X be the random variable defined for $x \in S$ by

$$X(x) = \begin{cases} 0 & \text{if } x \text{ is a vowel,} \\ 1 & \text{if } x \text{ is a consonant.} \end{cases}$$

Find $E(X)$ in the two cases, using probability P_1 and probability P_2 separately. Repeat for the random variable $Y(x) = i$ if x is the i-th letter of the alphabet.

Solution: $E(X) = 2/3$ and $E(X) = 1/2$ for P_1 and P_2, respectively, and $E(Y) = 1/6 + 2/6 + 3/6 + 4/6 + 5/6 + 6/6 = 21/6 = 7/2$ and $E(Y) = 1/4 + 2/8 + 3/8 + 4/8 + 5/4 + 6/8 = 27/8$ for P_1 and P_2, respectively.

□

2. *Exploring: Probability mass function, expectation, mean, variance.* Let the random variable X have the probability mass function

$$f(x) = \frac{(|x|+1)^2}{9}, \qquad x = -1, 0, 1.$$

(i) Recall that $f(x) = P(X = x)$. What is $P(X = 1)$? What is $P(X \geq 0)$?
Solution: $P(X = 1) = 4/9$, since this is $f(1)$. Now $P(X \geq 0) = P(X = 0) + P(X = 1) = f(0) + f(1) = 1/9 + 4/9 = 5/9$.

(ii) Compute $E(X), E(X^2)$, and $E(3X^2 - 2X + 4)$. Compute the variance σ^2 of the random variable X. Recall $\sigma^2 = E\left((X - \mu)^2\right)$ where $\mu = E(X)$.
Solution: First, $E(X) = -4/9 + 0 + 4/9 = 0$. Now $E(X^2) = \sum_{x \in S} x^2 f(x) = 8/9$, and so $E(3X^2 - 2X + 4) = 3E(X^2) - 2E(X) + E(4) = 3(8/9) - 2(0) + 4 = 20/3$. The variance is $\sigma^2 = E\left((X - 0)^2\right) = E(X^2) = 8/9$.

(iii) Repeat the problem for f defined by $f(-1) = \frac{2}{3}$, $f(0) = f(1) = \frac{1}{6}$.
Solution: With this p.m.f. we have $P(X = 1) = 1/6$ and $P(X \geq 0) = 1/3$. Also, $E(X) = -2/3 + 0 + 1/6 = -1/6$ and $E(X^2) = 2/3 + 0 + 1/6 = 5/6$ and $E(3X^2 - 2X + 4) = 3(2/3) - 2(-1/6) + 4 = 17/3$. Finally, the variance of X is $\sigma^2 = E\left((X - \mu)^2\right) = E\left((X + 1/6)^2\right)$, so

$$\begin{aligned}\sigma^2 &= (-1 + 1/6)^2(2/3) + (0 + 1/6)^2(1/6) + (1 + 1/6)^2(1/6) \\ &= 50/108 + 1/108 + 49/108 = 100/108 \\ &= 25/26.\end{aligned}$$

□

3. *Exploring: counting problems, conditional probabilities, independence, mutually exclusive events.* Un urn contains 10 balls: three of which are blue, and seven of which are green. Let an experiment consist of picking two balls, one at a time, from the urn without replacement. Describe the sample space of the experiment, and then find the probability of each possible outcome. Finally, let E be the event "the first ball chosen is blue" and F be the event "the second ball chosen is green".

Solution: The sample space could be defined to be $S = \{BB, BG, GB, GG\}$, for example. Then,

$$P(BB) = \frac{\binom{3}{2}}{\binom{10}{2}} = \frac{1}{15} \qquad P(BG) = \frac{1}{10} \cdot \frac{7}{9} = \frac{7}{30}$$

$$P(GG) = \frac{\binom{7}{2}}{\binom{10}{2}} = \frac{7}{15} \qquad P(GB) = \frac{7}{10} \cdot \frac{3}{9} = \frac{7}{30}$$

(i) Find $P(E)$, $P(F)$, and $P(E \cap F)$. Define *mutually exclusive.* Are E and F mutually exclusive?

Solution: $P(E) = P(BB) + P(BG) = 1/15 + 7/30 = 3/10$, $P(F) = P(GG) + P(BG) = 7/10$, and $P(E \cap F) = P(BG) = 7/30$. The events E and F are not mutually exclusive since two events are mutually exclusive if their intersection is empty, but $E \cap F = \{BG\}$.

(ii) Find $P(E|F)$. Are E and F independent events?

Solution: $P(E|F) = P(E \cap F)/P(F) = \frac{7/30}{7/10} = 1/3$. The events are NOT independent since $P(E|F) \neq P(E)$. In particular, knowing that the second ball chosen was green effects the probability that the first ball chosen was blue.

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