Math 3215 **Exam 1** Summer 2014

Instructor: Sal Barone (B)

Name: ________________________________

GT username: __________________________

1. No books or notes are allowed.
2. You may use ONLY NON-GRAPHING and NON-PROGRAMABLE scientific calculators. All other electronic devices are not allowed.
3. Show all work and fully justify your answer to receive full credit.
4. Circle or box your answers.
5. Good luck!

Page	Max. Possible	Points
1	30	
2	30	
3	40	
Total	100	

1. Throw two dice and let X be the sum of the numbers facing up. Describe the p.m.f. of X. Compute the mean μ of X and the variance σ^2 of X. (15 pts.)

Solution: The space of X is $S = \{2, 3, \ldots, 12\}$. The p.m.f. $f(x)$ of X is $f(x) = \frac{x-1}{36}$, $x = 2, \ldots, 7$, and $f(x) = \frac{13-x}{36}$ for $x = 8, \ldots, 12$, and note that $f(x) = f(14 - x)$, $x = 2, \ldots, 7$. The mean is

$$\begin{aligned} E(x) &= (2+12)\frac{1}{36} + (3+11)\frac{2}{36} + (4+10)\frac{3}{36} + (5+9)\frac{4}{36} + (6+8)\frac{5}{36} + 7\frac{6}{36} \\ &= \frac{14}{36}\sum_{x=1}^{5} x + \frac{7}{6} \\ &= \frac{7/18}{\cdot}15 + \frac{7}{6} \\ &= \boxed{7} \end{aligned}$$

The variance σ^2 is computed similarly:

$$\begin{aligned} E\left((X-\mu)^2\right) &= 2 \cdot 5^2 \cdot \frac{1}{36} + 2 \cdot 4^2 \cdot \frac{2}{36} + 2 \cdot 3^2 \cdot \frac{3}{36} + 2 \cdot 2^2 \cdot \frac{4}{36} + 2 \cdot \frac{5}{36} + 0 \\ &= \frac{1}{36}(50 + 64 + 54 + 32 + 10) \\ &= \frac{210}{36} = \boxed{\frac{35}{6}} \end{aligned}$$

□

2. A teacher is writing a multiple choice test for her class of 25 students and wants to give a different version of the test to each class, where each version has the exact same questions but are in a different order. What is the least number of questions the test must contain in order to accomplish this? *Justify your answer!* (15 pts.)

Solution: If she writes 4 questions then she can make at most $4! = 24$ tests, which is one too few. If she writes 5 questions then she could make $5! = 120$ tests, which is plenty. She needs to write at least 5 questions.

□

3. Let X be a uniformly distributed random variable on $S = \{1, 2, \ldots, m\}$. Find the value of m such that $E(X) = E\left((X-\mu)^2\right)$. *Hint:* $\sum_{x=1}^{m} x^2 = \frac{m(m+1)(2m+1)}{6}$. (15 pts.)

Solution: We begin by expanding $(X-\mu)^2$ and using the linearity of expectation, and the fact that $\mu = E(X)$.

$$E\left((X-\mu)^2\right) = E(X^2 - 2\mu X + \mu^2) = E(X^2) - 2\mu E(X) + E(\mu^2) = E(X^2) - \mu^2.$$

Now,

$$E(X^2) = \sum_{x=1}^{m} x^2 \frac{1}{m} = \frac{1}{m}\left(\frac{m(m+1)(2m+1)}{6}\right) = \frac{(m+1)(2m+1)}{6},$$

and

$$E(X) = \mu = \frac{m+1}{2}.$$

Combining all these we see that $E(X) = E\left((X-\mu)\right)$ is equivalent to

$$\frac{m+1}{2} = \frac{(m+1)(2m+1)}{6} - \left(\frac{m+1}{2}\right)^2,$$

dividing by the left hand side yields

$$1 = \frac{2m+1}{3} - \frac{m+1}{2}$$

and solving for m yields

$$m = 7.$$

Alternate proof: From class we have $\mu = \frac{m+1}{2}$ and $\sigma^2 = \frac{m^2-1}{12}$. Setting $\mu = \sigma$ and solving for m yields

$$\begin{aligned} \frac{m+1}{2} &= \frac{m^2-1}{12} \\ \implies \quad 6m + 6 &= m^2 - 1 \\ \implies \quad m^2 - 6m - 7 &= (m-7)(m+1) = 0 \end{aligned}$$

So $\boxed{m = 7}$ since $m = -1$ does not make sense.

□

4. On any given summer day in Atlanta, there is an 80% chance of it being sunny. I usually spend sunny days at the pool and there is a 30% chance that the day is both sunny and I'm at the pool. What is the chance I will go to the pool tomorrow, assuming that it will be sunny. (15 pts.)

Solution: Let E be the event "it is sunny" and F the event "at the pool". It is given that $P(E) = .8$ and $P(E \cap F) = .72$. We are asked to find $P(F|E) = \frac{P(E \cap F)}{P(F)} = \frac{.3}{.8} = \boxed{3/8 = 37.5\%}$.

□

5. In a random four digit number, what is the probability that the first digit is 3? What is the probability that the second digit is 3? Are these events mutually exclusive? Are these events independent? *You must justify your answer to the last questions for full credit.* (15 pts.)

Solution: The first digit is 3 10% of the time, as is the second digit. The first two digits are 3 with probability $100/10000 = 1/100$, and this is the product of the two probabilities $\frac{1}{10} \cdot \frac{1}{10} = \frac{1}{100}$, so the events are independent.

□

6. A policeman is pulling people over for speeding, but his radar gun isn't calibrated. If a car is speeding, the radar gun says "speeding" 90% of the time. However, the radar gun also claims that 5% of the people not speeding are actually speeding. Assume that 20% of the people on I-85 are speeding. If the policeman points his gun at a random car and it says "speeding" what is the probability that the car is actually speeding? (25 pts.)

Solution: Let E be the event "radar gun says speeding" and F be the event "the car is speeding". We want to find $P(F|E)$. We are given $P(E|F) = .9$ and $P(E|F^c) = .05$, so in particular, using e.g. Bayes's Theorem,

$$P(E) = P(F) \cdot P(E|F) + P(F^c)P(E|F^c) = .2(.9) + .8(.05) = .18 + .04 = .22.$$

Now $P(E \cap F) = P(F)P(E|F) = .2(.9) = .18$, and finally we have

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{.18}{.22} = \boxed{9/11 \approx 82\%}.$$

□