

## Practice Exam 2

1. What are the mean and variance of the Gamma distribution with parameters  $\alpha$  and  $\theta$ ?

*Solution:* The mean is  $\boxed{\mu = \alpha\theta}$  and the variance is  $\boxed{\sigma^2 = \alpha\theta^2}$ .

□

2. Describe in three sentences or less the relationship between the Exponential distribution, Poisson distribution, and a Poisson process.

*Solution:* The waiting times between successive changes in a Poisson process exhibit an exponential distribution, while the number of changes relates to the Poisson distribution.

□

3. Let  $X \sim N(1, 4)$ . Construct a new random variable  $Z$  from  $X$  that is standard normal,  $Z \sim N(0, 1)$ .

*Solution:* Define  $\boxed{Z = \frac{X - \mu}{\sigma}}$  to change a normally distributed continuous random variable  $X$  into a standard normal variable  $Z$ .

□

4. Let  $X$  and  $Y$  be continuous random variables that are uniformly distributed on  $[a, b]$  and  $[2a, 2b]$  respectively,  $0 < a < 1/2$  and  $5 < b$ . Which is greater  $P(X = 2)$  or  $P(Y = 2)$ ? Which is greater  $P(X \leq 2)$  or  $P(Y \leq 2)$ ?

*Solution:* Both  $P(X = 2) = P(Y = 2) = 0$ , so  $\boxed{\text{neither}}$  one is bigger. We have  $P(X \leq 2) = F(2) = \frac{2-a}{b-a}$  and  $P(Y \leq 2) = G(2) = \frac{2-2a}{2b-2a}$ . We see that

$$F(2) - G(2) = \frac{2-a}{b-a} - \frac{2-2a}{2b-2a} = \frac{2(2-a) - (2-2a)}{2(b-a)} = \frac{2}{2(b-a)} > 0,$$

so  $F(2) > G(2)$ , hence  $\boxed{P(X \leq 2) > P(Y \leq 2)}$ .

□

5. Let  $X \sim N(0, 1)$ . Find  $P(0.53 < X \leq 2.06)$  and  $P(|X| < 1.96)$ .

*Solution:* We have  $P(0.53 < X \leq 2.06) = P(X \leq 2.06) - P(X \leq 0.53) = .9803 - .7019 = \boxed{.2784}$ . Also,  $P(|X| < 1.96) = P(-1.96 < X < 1.96) = P(X < 1.96) - P(X \leq -1.96) = P(X < 1.96) - (1 - P(X \leq 1.96)) = 2P(X \leq 1.96) - 1 = 2(.975) - 1 = \boxed{.95}$ .

□

6. If the m.g.f. of  $X$  is  $M(t) = e^{166t+200t^2}$ , then find the mean and variance of  $X$  and  $P(170 \leq X < 200)$ .

*Solution:* There are two ways to solve this problem. The first is simply to compute

$$\begin{aligned} M'(t) &= (166 + 200t^2)e^{166t+200t^2} \implies \mu = M'(0) = 166, \\ M''(t) &= (166 + 400t)^2 e^{166t+200t^2} + 400e^{166t+200t^2} \implies M''(0) = 166^2 + 400, \\ \sigma^2 &= M''(0) - (M'(0))^2 = 166^2 + 400 - 166^2 = 400. \end{aligned}$$

The other is to note that  $X \sim N(\mu, \sigma^2)$  if the m.g.f. of  $X$  is of the form  $M(t) = e^{\mu t + \sigma^2/2t^2}$ , so in particular  $\boxed{X \sim N(166, 400)}$ .

□

7. The waiting times in minutes until two calls (*previously wrongly worded as “between successive calls”*) to 911 between noon and midnight yesterday were

20 28 81 4 9 41 9 11 10 24 20  
44 18 30 16 53 15 38 50 84 44 69

Suppose these calls are the result of sampling from a Gamma distribution with  $\alpha = 2$  and  $\theta = 120/7$ . Compare the theoretical distribution mean and variance with the mean and variance of the sample. Compare  $P(X < 35)$  with the proportion of times that are less than 35 minutes. Is the Gamma distribution a good model for this data?

*Solution:* We use google to compute the sample mean  $\bar{x}$  and variance  $s^2$  of these numbers and use  $\mu = \alpha\theta$  and  $\sigma^2 = \alpha\theta^2$  to obtain

$$\begin{aligned} \bar{x} &= 32.6 & s^2 &= 548.3, \\ \mu &= 34.3 & \sigma^2 &= 587.8. \end{aligned}$$

We are also asked to find  $P(X < 35)$ . We integrate,

$$\begin{aligned}
 P(X < 35) &= \int_0^{35} \frac{x^{2-1} e^{-7x/120}}{\Gamma(2) \left(\frac{120}{7}\right)^2} dx \\
 &= \int_0^{35} \frac{49x}{14400} e^{-7x/120} dx \\
 &= \frac{49x}{14400} \cdot \frac{120}{-7} e^{-7x/120} \Big|_0^{35} - \int_0^{35} \frac{49}{14400} \cdot \frac{120}{-7} e^{-7x/120} dx \\
 &= \frac{-7 \cdot 35}{120} e^{-7 \cdot 35/120} + \int_0^{35} \frac{7}{120} e^{-7x/120} dx \\
 &= (-49/24)(.1298) - e^{-7x/120} \Big|_0^{35} \\
 &= (-49/24)(.1298) - (.1298 - 1) = -.2650 + .8702 = \boxed{.6052}
 \end{aligned}$$

The relative frequency of this event is  $13/22 \approx \boxed{.59}$ .

It seems the Gamma distribution is a good model for this data. For a gamma distribution for  $\alpha = 2$  we expect to see numbers which correspond to the waiting time until the second call comes in.

□

8. A kid loves prizes at the bottom of cereal boxes and is trying to get all four prizes from his favorite cereal. The four different prizes are randomly put into the boxes at the factory. If his mom decides to buy the cereal until all four prizes are obtained, then what is the expected number of boxes she will have to buy until the boy has obtained all four prizes?

*Solution:* We can think of this as four Poisson process, one after the other. Notice that when he first opens a box, there is certainly one of the four prizes inside, so he has a 100% chance to get a prize he doesn't already have. Afterwards, there is a diminishing chance that he gets a *new* prize each time he finds a new prize. In particular, after opening the first box, there is a 75% chance that subsequent boxes will contain a new prize. After opening the box with the 2nd new prize, there is a 50% chance that the subsequent boxes will contain a new prize, and finally after opening the box with the 3rd new prize, there is only a 25% chance that a subsequent box will contain a new prize. Let  $X_1, X_2, X_3, X_4$  be the discrete random variables

$X_1$  = number of box containing first new prize,

$X_2$  = number of additional boxes until obtaining second new prize,

$X_3$  = number of additional boxes until obtaining third new prize,

$X_4$  = number of additional boxes until obtaining fourth new prize.

Then, each random variable  $X_i$  has a geometric distribution corresponding to a sequence of Bernoulli trials with probability of success  $p_i = (5 - i)/4$ ,  $i = 1, 2, 3, 4$ . So in particular, if  $X = X_1 + X_2 + X_3 + X_4$  is the total number of boxes you need, from the linearity of expectation we have  $E(X) = E(X_1) + E(X_2) + E(X_3) + E(X_4) = 1 + 4/3 + 4/2 + 4/1 = 8.33$ . She will likely have to buy  $\boxed{9}$  boxes of the cereal.

□

9. Let  $X$  have a Poisson distribution with variance  $\sigma^2 = 3$ . Find  $P(X = 2)$ .

*Solution:* We have  $\mu = \lambda = \sigma^2 = 3$ . So  $f(x) = \frac{3^x e^{-3}}{x!}$ . In particular,  $P(X = 2) = f(2) = \frac{3^2}{2!e^3} \approx \boxed{.075}$ .

□