Practice Exam 3: Solutions

1. Consider the function $f(x,y) = \alpha(x+2y)$ if $0 < x, y < 1$ and $f(x,y) = 0$ otherwise. Find a value α such that f is a joint p.d.f. Is this choice unique? Find the marginal p.d.f.'s of X and Y. Find the covariance of X and Y. Is it possible to determine whether or not X and Y are independent **only** using what you just found out about the covariance?

Solution: The space of X and Y is $S = \{(x,y) : \ 0 < x < 1, 0 < y < 1\}$ and we need $\iint_S f(x,y)\, dydx = 1$. Integrating, we have

$$\begin{aligned}\iint_S f(x,y)\, dydx &= \int_0^1 \int_0^1 \alpha(x+2y)\, dydx \\ &= \alpha \int_0^1 (xy + y^2)\big|_0^1\, dx \\ &= \alpha \int_0^1 (x+1)\, dx = \alpha\, (x^2/2 + x)\big|_0^1 = 3\alpha/2.\end{aligned}$$

So, $\alpha = 2/3$. The marginal p.d.f.'s of X and Y are found via integrating:

$$\begin{aligned}f_1(x) &= \int_0^1 f(x,y)\, dy = \int_0^1 .67(x+2y)\, dy = .67xy + .67y^2\big|_0^1 = .67x + .67, \\ f_2(y) &= \int_0^1 f(x,y)\, dx = \int_0^1 .67(x+2y)\, dx = .67x^2 + 1.33xy\big|_0^1 = .67 + 1.33y.\end{aligned}$$

Recall $\text{Cov}(X_1, X_2) = E((X_1 - \mu_1)(X_2 - \mu_2)) = E(X_1X_2) - \mu_1\mu_2$, so to find the covariance $\text{Cov}(X,Y)$ we will first compute $E(XY), \mu_1, \mu_2$. We have

$$\begin{aligned}E(X_1X_2) &= \int_0^1 \int_0^1 .67xy(x+2y)\, dydx = \int_0^1 \int_0^1 .67x^2y + 1.33xy^2\, dydx = .333 \\ \mu_1 &= \int_0^1 .67x^2 + .67x\, dx = .55833 \\ \mu_2 &= \int_0^1 .67y + 1.33y^2\, dy = .77833\end{aligned}$$

So, $\text{Cov}(X,Y) = (1/3) - (5/9)(11/18) = -1/162 = -.0062$. We know that if X_1 and X_2 are independent, then $\text{Cov}(X_1, X_2) = 0$. Thus, X and Y must not be independent, since $\text{Cov}(X_1, X_2) \neq 0$. That is, $\text{Cov}(X,Y) \neq 0 \implies X$ and Y are independent.

□

2. State the Central Limit Theorem. A soda company has installed a new machine to fill its bottles. The machine was poorly installed, so the amount the machine dispenses has a **variance** *(changed from standard deviation)* of 0.1 liters, and the mean is 2 liters. Approximate the probability that after 40 bottles are filled that the the sample mean is at least 1.9 liters.

Solution: The Central Limit Theorem says that if $\bar{X}$ is the mean of a random sample of size n coming from a distribution with mean μ and variance σ^2, and $Z = (\bar{X} - \mu)/(\sigma/\sqrt{n})$, then Z is approximately standard normal $N(0,1)$, and that the approximation is better for large values of n.

We set $Z = (\bar{X} - 2)/(\sqrt{.1/40})$ and then $P(\bar{X} \geq 1.9) = P(Z \geq 2) \approx 1 - .9772 = \boxed{.0228}$.

□

3. The cumulative distribution function of a continuous random variable is given by $F(x) = 1 - e^{-3x}$, $x > 0$. Find the p.d.f. of this random variable.

Solution: The p.d.f. is $F'(x) = f(x) = 3e^{-3x}$, $x > 0$.

□

4. Cars arrive at a toll booth at a rate of 4 calls every 6 minutes. Assume these cars arrive as a Poisson process. What is the probability that the 5th car arrives at exactly 6 minutes and 45 seconds?

Solution: The probability that the cars arrive at *exactly* any time is zero.

□

5. Let X and Y be random variables on the space $S = \{(0,0),(1,1),(1,-1),(2,0)\}$ with joint p.m.f. $f(x,y) = 1/4$. Compute the covariance and correlation coefficient. Are X and Y independent? Can you tell if they are independent **only** from the correlation coefficient?

Solution: We compute $\mathrm{Cov}(X,Y) = E(XY) - \mu_1\mu_2$ and $\rho = \mathrm{Cov}(X,Y)/(\sigma_1\sigma_2)$. We have,

$$\begin{aligned} \mu_1 &= \sum_S x f(x,y) = (0+1+1+2)/4 = 1, \\ \mu_2 &= \sum_S y f(x,y) = (0+1-1+0)/4 = 0, \\ E(XY) &= \sum_S xy f(x,y) = (0+1-1+0)/4 = 0. \end{aligned}$$

So, $\mathrm{Cov}(X,Y) = 0 - 1 \cdot 0 = 0$ and $\rho = 0/(\sigma_1\sigma_2) = 0$. It is not possible to tell if X and Y are independent simply by knowing that $\mathrm{Cov}(X,Y) = 0$. In fact, by observing that $X = 0$ implies $Y = 0$, we see that X and Y are clearly not independent (but from class if $\mathrm{Cov}(X,Y) = 0$ implies that X and Y are independent).

□

6. Let $X_1, \ldots, X_8$ be a random sample from a distribution having p.m.f. $f(x) = (x+1)/6$, $x = 0,1,2$. What is the p.m.f. of $Y_1 = X_1 + X_2$? What is the p.m.f. of $Y_2 = X_3 + X_4$? What about $Y = X_1 + X_2 + X_3 + X_4$ and $W = X_1 + \ldots + X_8$?

Solution: We have that the p.m.f. of $Y_1 = X_1 + X_2$ is $g(y) = \sum_{i=0}^{4} f(i)f(y-i) = f(0)f(y) + f(1)f(y-1) + \ldots f(4)f(y-4)$, and note that many of these terms are often zero. In particular,

$$\begin{aligned} g(0) &= f(0)f(0) = 1/36 \\ g(1) &= f(0)f(1) + f(1)f(0) = 4/36 \\ g(2) &= f(0)f(2) + f(1)f(1) + f(2)f(0) = (3+4+3)/36 = 10/36 \\ g(3) &= f(1)f(2) + f(2)f(1) = 12/36 \\ g(4) &= f(2)f(2) = 9/36. \end{aligned}$$

Clearly, $Y_2 = X_3 + X_4$ has the same p.m.f. as that of Y_1 and they are independent random variables. So $Y = X_1 + X_2 + X_3 + X_4 = Y_1 + Y_2$ has the p.m.f. $h(y) = \sum_{i=1}^{8} g(i)g(y-i)$, where again many of the terms might be zero depending on the choice of $y \in \{0, \ldots, 8\}$. We have

$$\begin{aligned}
h(0) &= g(0)g(0) = 1/6^4 \\
h(1) &= g(0)g(1) + g(1)g(0) = 8/6^4 \\
h(2) &= g(0)g(2) + g(1)g(1) + g(2)g(0) = (10 + 16 + 10)/6^4 = 36/6^4 \\
h(3) &= g(0)g(3) + g(1)g(2) + g(2)g(1) + g(3)g(0) = (2 \cdot 12 + 2 \cdot 40)/6^4 = 104/6^4 \\
h(4) &= g(0)g(4) + g(1)g(3) + g(2)g(2) + g(3)g(1) + g(4)g(0) = (2 \cdot 9 + 2 \cdot 48 + 10)/6^4 = 124/6^4 \\
h(5) &= g(1)g(4) + g(2)g(3) + g(3)g(2) + g(4)g(1) = (2 \cdot 36 + 2 \cdot 120) = 312/6^4 \\
h(6) &= g(2)g(4) + g(3)g(3) + g(4)g(2) = (2 \cdot 90 + 12)/6^4 = 192/6^4 \\
h(7) &= g(3)g(4) + g(4)g(3) = 216/6^4 \\
h(8) &= g(4)g(4) = 81/6^4
\end{aligned}$$

Clearly, the random variable $Y' = X_5 + X_6 + X_7 + X_8$ has the same p.m.f. as that of Y, and Y, Y' are independent. So, the p.m.f. of $W = X_1 + X_2 + \ldots + X_8 = Y + Y'$ is $p(w) = \sum_{i=1}^{16} h(i)h(w-i)$.

We can also unwrap all the definitions to write the p.m.f. of W as follows.

$$\begin{aligned}
p(w) &= \sum_{i=1}^{16} h(i)h(w-i) \\
&= \sum_{i=1}^{16} \left(\sum_{j=1}^{8} g(j)g(i-j) \right) \left(\sum_{k=1}^{8} g(k)g(w-i-k) \right) \\
&= \sum_{i=1}^{16} \sum_{j=1}^{8} \sum_{k=1}^{8} g(j)g(i-j)g(k)g(w-i-k) \\
&= \sum_{i=1}^{16} \sum_{j,k=1}^{8} \sum_{l,m,n,o=1}^{4} f(l)f(j-l)f(m)f(i-j-m)f(n)f(k-n)f(o)f(w-i-k-o)
\end{aligned}$$

□