Homework 1: Due 5/22/14

1. A four sided die is rolled 25 times and the following numbers are recorded:

$$4, 3, 2, 3, 1, 3, 1, 2, 1, 1, 3, 4, 2, 4, 1, 2, 2, 4, 4, 1, 2, 2, 3, 3, 4.$$

Construct a table showing the frequency and relative frequency of each possible result, then sketch a relative frequency histogram (you are welcome to use computer software or sketch by hand). What is the sample space of this experiment? What is the random variable?

Solution: The sample space of the experiment "roll a four sided die" is $\{1, 2, 3, 4\}$ and the random variable is $X =$ number facing upwards.

X	frequency	relative frequency
1	6	24%
2	7	28%
3	6	24%
4	6	24%

$\square$

2. Prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Solution: We will use the inclusion-exclusion principle

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

First,
$$P(A \cup (B \cup C)) = P(A) + P(B \cup C) - P(A \cap (B \cup C)),$$

and also

$$P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C))$$
$$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap A \cap C), \text{ and}$$
$$P(B \cup C) = P(B) + P(C) - P(B \cap C).$$

Substitution of the second and third equations into the first we get the desired result

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C).$$

$\square$

3. When getting a physical exam, the probability of having neither lab work nor a referral to a specialist is 0.23. The probability of having lab work is 0.41 and the probability of getting a referral is 0.52. What is the probability of getting lab work but not referral work? If the probabilities above are instead relative frequencies in a sample of 120 patients, then how many patients had both lab work and a referral?

Solution: Consider the random experiment described above. The sample space of the random experiment has four elements corresponding to the two binary choices: having lab work and getting a referral. Consider the events

$$A : \text{lab work is done,}$$
$$B : \text{referral work is done.}$$

We want $P(A \cap B^c)$ and we know $P(A^c \cap B^c) = 0.23$, $P(A) = .41$ and $P(B) = .52$. Since $(A \cup B)^c = A^c \cap B^c$ we have $P(A \cup B) = 1 - .23 = .77$. Next, $P(A \cap B) = P(A) + P(B) - P(A \cup B) = .41 + .52 - .77 = .16$ and finally $P(A \cap B^c) = P(A) - P(A \cap B) = .41 - .16 = .25$. So the probability $\boxed{P(A \cap B^c) = 25\%}$. Since we already computed $P(A \cap B) = 16\%$, the frequency of the event $A \cap B$ in 120 trials is $\boxed{.16 * 120 = 19.2}$.

□

4. You want a pizza but can't decide what to order. The pizza comes in three sizes small, medium, and large; white or wheat dough; and there are 9 toppings you can choose from (you can also choose to have no toppings). How many different pizza combinations could you make?

Solution: The three stage experiment consists of choosing one of three sizes, a binary choice of dough, and to select a subset of 9 toppings. Thus by the multiplication principle the number of pizzas is $\boxed{3 \cdot 2 \cdot 2^9 = 3072}$.

□

5. The Georgia Lottery is considering a game where 4 numbers are randomly drawn (with replacement) between 0 and 9. You win a prize if any permutation of your selected integers is chosen. What are the odds of winning if you select four different numbers? 3 numbers with one repeated? Two pairs of numbers?

Solution: Suppose you select $1, 2, 3, 4$. There are $10^4 = 10,000$ strings of four digit strings (allowing repeats) and of these $4! = 24$ consist of the digits $1, 2, 3, 4$. So the probability of

winning if you choose $1, 2, 3, 4$ is $\boxed{24/10,000 = .24\%}$. Clearly any other string of four distinct digits has the same probability (there is nothing special about choosing these particular four).

If you select $1, 2, 3, 3$, then there are $4!/2 = 12$ different four digit strings out of 10,000 where you win, so the probability of winning is $\boxed{.12\%}$. If you select $1, 1, 3, 3$ then your chances are even worse because there are only $4!/4 = 6$ ways for you to win, so the probability of winning is only $\boxed{.06\%}$.

$\square$

6. Find the probability of the event "exactly three heads occur" in an experiment where a fair coin is flipped 5 times.

   Solution: Exactly three heads occur in $\binom{5}{3} = 10$ outcomes and there are a total of $2^5 = 32$ outcomes. So the probability is $\boxed{10/32 = 31.25\%}$.

   $\square$

7. A hat is filled with 5 slips of paper, one saying *win* and four that say *lose*. You and another payer take turns picking pieces of paper until someone wins. If you draw first, find the probability of winning if the sampling is done with replacement. What about without replacement?

   Solution: This has been explained very well on Piazza, but you compute the probabilities

   $$P(\text{1rst round}|\text{win}) = \frac{1}{5},$$

   $$P(\text{2rst round}|\text{win}) = \frac{1}{5}\left(\frac{4}{5}\right)^2,$$

   $$P(\text{3rst round}|\text{win}) = \frac{1}{5}\left(\frac{4}{5}\right)^4,$$

   $$\vdots$$

   $$P(n\text{th round}|\text{win}) = \frac{1}{5}\left(\frac{4}{5}\right)^{2(n-1)}.$$

   and note that the event $\{\text{win}\}$ is the disjoint union of the events $\{\text{win in round } n\}$, $1 \leq n$. So

   $$P(\text{win}) = \sum_{n=1}^{\infty}\frac{1}{5}\left(\frac{4}{5}\right)^{2(n-1)} = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{16}{25}\right)^n = \frac{1}{5}\cdot\frac{1}{1-\frac{16}{25}} = \boxed{5/9}.$$

   $\square$

8. A student is asked to write down a string of one-digit numbers between 0 and 9 that seem to be random. She writes down the following string of 20 numbers:

$$8, 2, 6, 0, 3, 9, 1, 6, 5, 8, 7, 4, 9, 5, 0, 5, 2, 7, 5, 2.$$

Are the numbers really random? (You don't have to answer that one, that was rhetorical) In a random 20 digit string of numbers, what is the probability that 2 adjacent numbers are the same? that they differ by one (consider 9 and 0 to differ by one)? that they differ by more than one? What is the relative frequency of these three events in the sample above? In the sample, which of the events is *more likely than random*? less likely than random?

Solution: The probability that two adjacent numbers are the same is 10%, that they differ by one is 20% and that they differ my more than one is 70%. The associated frequencies are $0, 2, and 17$ (by the way they have to add up to 19, do you see why?). The associated relative frequencies are $0\%, 10.53\%$, and 89.47%. The first two are lower than expected by about 10% and the last is higher than expected by about 20%.

$\square$