Homework 3: Due 6/12/14

1. *You do not need to justify your answer to this question.* List, without proof, the p.m.f., mean, and variance of the random variable X where X has the following distributions: negative binomial, geometric, Poisson.

Solution: For negative binomial with parameter r, probability of success in one Bernoulli trial p and X is the number of succeses we have:

$$f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r},\ \mu = r/p,\ \sigma^2 = r(1-p)/p^2.$$

For geometric, set $r = 1$ in the above to obtain $\boxed{f(x) = pq^{x-1},\ \mu = 1/p,\ \sigma^2 = q/p^2}$, where $q = 1 - p$. For Poisson distribution, we have

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!},\ \mu = \sigma^2 = \lambda.$$

□

2. An urn contains 6 white balls and 9 red balls. Pick a ball at random from the urn. Let X be the random variable $X = 0$ if the ball is white and $X = 1$ if the ball is red. Find the p.m.f. of X, and μ, σ^2 of X.

Solution: The random variable X is the random variable of a Bernoulli trial. We have $E(X) = 0(6/15) + 1(9/15) = 9/15 = \boxed{3/5}$, and $\sigma^2 = E(X^2) - \mu^2 = (3/5) - (3/5)^2 = (15-9)/25 = \boxed{6/25}$. The p.m.f. is

$$f(x) = \begin{cases} 6/15 & \text{if } x = 0 \\ 9/15 & \text{if } x = 1. \end{cases}$$

□

3. A company that manufactures 500 foot long wires knows that any particular 20 foot section of wire has on average one defect. Find the probability that a 60 foot section of wire has no defects. What is the probability that a 60 foot section has at most 2 defects? (You may use the table on the website to answer this question if you desire)

Solution: The random variable X which counts the number of defects in a 60 foot section of wire follows a Poisson distribution with $\lambda = (1/20 \text{ defect/foot}) \cdot (60 \text{ feet}) = 60/20 = 3$. So we have $f(x) = \frac{3^x e^{-3}}{x!}$. We are asked to find $f(0) = P(X = 0)$ the probability that there are no defects, so we have $f(0) = e^{-3} \approx \boxed{.05}$. Also, we want to find $P(X \leq 2) = f(0) + f(1) + f(2)$, which is easily computed by hand

$$P(X \leq 2) = e^{-3}(1 + 3 + 9/2) = 17e^{-3}/2 \approx \boxed{.42}.$$

□

4. A newspaper stand orders only 3 copies of a certain newspaper because the manager knows that this particular paper is not purchased very often. If the number of purchases of this particular paper per day follows a Poisson distribution with $\mu = 2$, then how many of these newspapers does the stand sell on average in a day? How many papers should the manager buy so that the chance of someone not being able to purchase the paper on any given day is less than 5%?

Solution: For a Poisson distribution $\lambda = \mu$ so we have the probability mass function of X the number of purchases per day is given by

$$f(x) = \frac{2^x e^{-2}}{x!}.$$

We are asked to find the number x such that $P(X > x)$ is at most 5%, and note that $P(X > x) = 1 - P(X \leq x)$. So,

$$P(x > x) > .05 \quad \Longleftrightarrow \quad 1 - P(X \leq x) > .05 \quad \Longleftrightarrow \quad P(X \leq x) < .95.$$

So, we will find the number x such that $P(X \leq x) < .95$ (but there are likely other ways to approach this problem). One way to find this value is just guess and check: one can compute $P(X \leq x) \sum_{0 \leq i \leq x} f(i)$ for various values of x until one exceeds .95. Alternately, one can look up in the Table III of these vales. In either case, it doesn't take very long to notice that $P(X \leq 4) = .947$ and $P(X \leq 5) = .983$, so he should order 2 additional papers per day for a total of $x = 5$.

□

5. Find c such that $f(x) = cx^{-2}$, $1 \le x \le \infty$ is a probability density function of a continuous random variable X. What is the expected value $E(X)$?

Solution: We just need to pick c such that

$$P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x)\, dx = 1.$$

Immediately we see that

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{1}^{\infty} cx^{-2}\, dx = \quad -cx^{-1}\Big|_{1}^{\infty} = 0 - (-c) = c.$$

So, we need $\boxed{c = 1}$. In this case, $E(X) = \int_{-\infty}^{\infty} xf(x)\, dx$ so

$$E(X) = \int_{1}^{\infty} x \cdot x^{-2}\, dx = \int_{1}^{\infty} \frac{1}{x}\, dx = \quad \ln(x)\Big|_{1}^{\infty} = \lim_{n\to\infty} \ln(n),$$

which does not exist. So $E(X)$ is not defined. $\square$

6. Find d such that $f(x) = dx^{-3}$, $1 \le x \le \infty$ is a p.d.f. of a continuous random variable X. What is $E(X)$? What is the variance $Var(X)$?

Solution: Similar to above.

$$\int_{1}^{\infty} dx^{-3}\, dx = \quad \tfrac{-d}{2} x^{-2}\Big|_{1}^{\infty} = 0 - \tfrac{-d}{2} = \tfrac{d}{2}.$$

So $\boxed{d = 2}$. In this case

$$E(X) = \int_{1}^{\infty} 2x \cdot x^{-3}\, dx = \int_{1}^{\infty} 2x^{-2}\, dx = \quad -2x^{-1}\Big|_{1}^{\infty} = 2,$$

but it is easily checked and similar to the last problem that that $E(X^2)$ does not exist, so $Var(X)$ does not exist. $\square$

7. The demand X for gas at a gas station has the p.d.f. $f(x) = 4x^3e^{-x^4}$, $0 < x < \infty$, where x is in thousands of gallons. If the gas station manager only has 1000 gallons of gas at the beginning of the week, what is the probability that the station runs out of gas at some point during the week?

Solution: We want $P(X > 1)$ since X was already in thousands. We compute

$$P(X > 1) = 1 - P(X \le 1) = 1 - F(1) = 1 - \int_{0}^{1} 4x^3e^{-x^4}\, dx = 1 - \quad \text{-}e^{x^4}\Big|_{0}^{1} = 1 + (1/e) - 1 = \boxed{1/e}.$$

$\square$