Homework 3b: Due 6/19/14

1. Define what it means for X to be a continuous random variable. How is the probability density function $f(x)$ used to calculate $P(X \leq x)$? What does the distribution function $F(x)$ measure?

Solution: A random variable X is a continuous random variable X if it takes values in $\mathbb{R}$, not just values in some discrete set such as $\mathbb{N}$ or $\mathbb{Z}$. The p.d.f. $f(x)$ is used to calculate $P(X \leq x)$ via integration, $P(X \leq x) = \int_a^x f(t)\,dt$, where the space of X is (a, b) and $a, b \in \mathbb{R} \cup \{\pm\infty\}$. The distribution function $F(x) = P(X \leq x)$ measures the probability that X is less than x (or equal to x, since integrating over a single point yields zero).

□

2. Cars arrive randomly at a 200 second stoplight (*i.e.*, a stoplight which is red for 200 seconds). Let X be the amount of time a randomly selected car has to wait at the light before it turns green. If X is $U(0, 200)$, meaning that it is uniformly distributed on the interval $[0, 200]$, find the p.d.f. of X, and find the probability that the car must wait longer than 2 minutes. What is the probability that the car has to wait between 1 and 2 minutes at the light?

Solution: If X is $U(0, 200)$ distributed continuously and uniformly on $[0, 200]$ then $f(x) = \frac{1}{b-a} = \frac{1}{200}$, since $f(x)$ must be constant and with this p.d.f. $\int_0^{200} \frac{1}{200}\,dt = 1$. Note that a simple calculation shows that the mean is $\mu = \frac{a+b}{2} = \frac{0+200}{2} = 100$ the midpoint of the interval $[0, 200]$, and the variance of X is $\sigma^2 = \frac{(b-a)^2}{2} = 20,000$.

The probability that the car has to wait longer than 2 minutes is $P(X > 120) = P(120 < X < 200) = \int_{120}^{200} \frac{1}{200}\,dt = (200 - 120)/200 = 40\%$.

The probability that the car has to wait between 1 and 2 minutes is $P(60 < X < 120) = \int_{60}^{120} \frac{1}{200}\,dt = (120 - 60)/200 = 30\%$.

□

3. Suppose the lifespan of a certain type of electrical component follows an exponential distribution with a mean life of 50 days. If X denotes the life of this component (in days) then find $P(X > x)$, which is a function of x the number of days before failure. Find $P(X > 20)$ and also find the conditional probability $P(X > 40|X > 20)$, the probability that the component lasts 40 days given that it lasts 20 days. Are these probabilities equal? Is an exponential a good model for the lifespan of a component?

Solution: We have that the p.d.f. of the lifespan X in days is $f(x) = \frac{1}{50}e^{-x/50}$, and hence

$$P(X > x) = 1 - P(X \leq x) = 1 - \int_0^x \frac{1}{50}e^{-t/50}\,dt = e^{-x/50}.$$

Hence $P(X > 20) = e^{-20/50} \approx .6703$. We have

$$P(X > 40|X > 20) = \frac{P(X > 40)}{P(X > 20} = \frac{e^{-40/50}}{e^{-20/50}} = e^{-20/50} \approx .6703.$$

It seems that the exponential distribution is not a good model for the lifespan of a component since you should expect each span of 20 days to have a greater likelihood of failure than the previous span.

□

4. If 10 observations are taken independently from a chi-square distribution with 19 degrees of freedom, find the probability that exactly 2 of the 10 sample items exceed 30.14.

Solution: Let X be the value of the observation. From Table IV we have that $P(X > 30.14) = .05$. So, the probability that 2 of the 10 sample items exceed 30.14 is exactly

$$\binom{10}{2}(.05)^2(.95)^8 = \boxed{.0746}.$$

□

5. Cars arrive at a toll booth at a mean rate of three cars every 4 minutes according to a Poisson process. What is the probability that there are fewer than two cars in a 4 minute period? Find the probability that the toll booth collector has to wait longer than 10 minutes to collect the 9th toll.

Solution: There are several ways to solve this problem, either by thinking of the waiting time or the number of occurrences in an interval (I'll do one of each). Let X be the waiting time until the 2nd car enters the toll booth. Then X follows a gamma distribution with $\alpha = 2$, has p.d.f.

$$f(x) = \frac{16}{9} x e^{-4x/3}, \qquad 0 \leq x < \infty, \quad \theta = \frac{3}{4}, \quad \mu = \frac{3}{2}.$$

We have that the probability that less than 2 cars enter the toll booth in the first 4 minutes is $P(X > 4)$, since in this case the waiting time for the second car is more than 4 minutes, and $P(X > 4) = 1 - \int_0^4 \frac{16}{9} x e^{-4x/3}\, dx$. We solve this integral using integration by parts.

$$\begin{aligned}
\int_0^4 \tfrac{16}{9} x e^{-4x/3}\, dx &= \tfrac{-4x}{3} e^{-4x/3}\Big|_0^4 - \int_0^4 \tfrac{-4}{3} e^{-4x/3}\, dx \\
&= \tfrac{-16}{3} e^{-16/3} - e^{-4x/3}\Big|_0^4 \\
&= \tfrac{-16}{3} e^{-16/3} - e^{-16/3} + 1.
\end{aligned}$$

Hence, $P(X > 4) = 1 - (\frac{16}{3} e^{-16/3} - e^{-16/3} + 1) \approx .031$.

Now let Y be the number of occurrences (cars at the toll booth) in a 10 minute span of time. Then Y follows the Poisson distribution with mean $\mu = \frac{3}{4} \cdot 10 = 7.5$, and hence the p.d.f. of Y is $g(y) = \frac{(7.5)^x e^{-7.5}}{x!}$. We are asked to find the probability that the 9th toll takes longer than 10 minutes to collect, which is $P(Y \leq 9) \approx .776$ using Table III.

□