Homework 5b: Due 7/10/14

1. Let X_1, X_2, X_3 be a random sample from a distribution with p.d.f. $f(x) = 2e^{-2x}$, $0 < x < \infty$. This means that X_1, X_2, X_3 are independent random variables each with the same p.d.f. $f(x)$, or said even another way that they are independent identically distributed random variables (written i.i.d.). Find the probability $P(0 < X_1 < 2, 2 < X_2 < 4, 4 < X_3 < 6)$. What is the joint p.d.f. of X_1, X_2, X_3? What is the probability that exactly one of X_1, X_2, X_3 is in the range $0 < x < 2$ and exactly one is in the range $2 < x < 4$ and exactly one is in the range $4 < x < 6$?

Solution: We have that the joint p.d.f. of X_1, X_2, X_3 is $f(x_1, x_2, x_3) = f(x_1)f(x_2)f(x_3) = 2^3 e^{-2(x_1+x_2+x_3)}$, since the random variables are independent. Hence,

$$P(0 < X_1 < 2, 2 < X_2 < 4, 4 < X_3 < 6) = \int_0^2 \int_2^4 \int_4^6 2^3 e^{-2(x_1+x_2+x_3)}\, dx_3 dx_2 dx_1 = \boxed{5.8 \times 10^{-6}}.$$

To find the probability that exactly one variable is in the specified range, we multiply by $3! = 6$, which is the number of permutations of the three variables, and hence get that this probability is $\boxed{3.49 \times 10^{-5}}$.

□

2. Let X_1, X_2 be independent random variables with respective binomial distributions $b(3, .25)$ and $b(4, .5)$. Find $P(X_1 = 2, X_2 = 3)$ and $P(X_1 + X_2 = 5)$.

Solution: Since X_1, X_2 are independent, the joint p.m.f. $f(x_1, x_2) = f_1(x_1)f_2(x_2)$, where $f_1(x_1) = (.25)^{x_1}(.75)^{1-x_1}$ and $f_2(x_2) = (.5)^{x_2}(.5)^{1-x_2}$. We have $f_1(2) = .0469$ and $f_2(x_2) \equiv (.5)^4 = .0625$. So $f(2, 3) = \boxed{.0029}$.

□

3. Let X_1, X_2, X_3 be i.i.d. random variables with Poisson distributions with mean $\lambda = 3$. Find the moment generating function of $Y = X_1 + X_2 + X_3$. How is Y distributed?

Solution: We use the fact from class that the moment generating function $M(t)$ of $Y = X_1 + X_2 + X_3$ is the product of the moment generating functions of X_1, X_2, X_3, so

$$M(t) = \left(e^{3(e^t-1)}\right)^3 = e^{9(e^t-1)}.$$

Since $M(t)$ is the moment generating function of a Poisson distribution with $\lambda = 9$, this is distribution of Y.

□

4. Let $\bar{X}$ denote the mean of a random sample of size 25 from a distribution whose p.d.f. is $f(x) = x^3/4$, $0 < x < 2$. It is easy to show that $\mu = 8/5$ and $\sigma^2 = 8/75$. Use the central limit theorem to approximate $P(1.4 < \bar{X} < 1.7)$.

Solution: From the central limit theorem, the random variable $\bar{X}$ is approximately normally distributed with mean $\mu =$ and variance $\sigma^2/n = .0043$. Hence, the standard deviation of $\bar{X}$ is .0653. We define the (approximately) standard normal variable $Z = (\bar{X} - 1.6)/.0653$ and look up in the table

$$\begin{aligned} P(1.4 < \bar{X} < 1.7) &= P\left(\frac{1.4 - 1.6}{.0653} < Z < \frac{1.7 - 1.6}{.0653}\right) \\ &\approx P(-3.0628 < Z < 1.534) \\ &= P(Z < 1.534) - P(Z < -3.0628) \\ &= .9382 - (1 - .9989) = \boxed{.9371}. \end{aligned}$$

□

5. Let X_i, $1 \le i \le n$ be a random sample of size n from the continuous uniform distribution $U(0, 1)$ with p.d.f. $f(x) = 1$. Find the mean μ_i and variance σ_i^2 of X_i, $1 \le i \le n$. Find the mean and variance of $\bar{X} = \frac{X_1 + \cdots + X_n}{n}$. Approximate $P(\bar{X} \le n/2)$ using the central limit theorem.

Solution: The mean of X_i is $\mu_i = \frac{a+b}{2} = .5$ for $i = 1, \ldots, n$, and the mean of $\bar{X}$ is also $\mu = \sum_{i=1}^{n} \frac{1}{n}.5 = .5$. The variance of X_i is $\sigma_i^2 = \frac{(b-a)^2}{12} = 1/12$ for $i = 1, \ldots, n$, and the variance of $\bar{X}$ is $\sum_{i=1}^{n} \frac{1}{n^2} 1/12 = 1/(12n)$. Now, $\bar{X}$ is approximately normal $N(.5, 1/(12n))$, so the random variable $Z = (\bar{X} - \mu)/\sigma$ is approximately standard normal $N(0, 1)$. Hence,

$$\begin{aligned} P(\bar{X} \le n/2) &= P(Z \le (n/2 - .5)/(1/\sqrt{12n})) \\ &P(Z \le \sqrt{12n}(n/2 - 1/2)) \\ &P(Z \le \sqrt{3}(n^{1.5} - n^{.5})) \end{aligned}$$

□