Group Quiz 6

You may work in groups, ask the instructor for the help, use books or notes or the internet. Any calculator can be used on this quiz.

1. Let W_1, W_2 be independent random variables with the *Cauchy distribution* defined by having the *Cauchy p.d.f.*

$$h(w) = \frac{1}{\pi(1+w^2)}, \qquad -\infty < x < \infty.$$

This problem will help us to find the p.d.f. of the sample mean $Y = (W_1 + W_2)/2$ using convolutions. First, by doing a change of variables show that the p.d.f. of $X_i = W_i/2$, $i = 1, 2$, is

$$f(x) = \frac{2}{\pi(1+4x^2)}, \qquad -\infty < x < \infty.$$

Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1$ and show that the joint p.d.f. of Y_1, Y_2 is

$$g(y_1, y_2) = f(y_1 - y_2)f(y_2), \qquad -\infty < y_1, y_2 < \infty.$$

Finally show that

$$g_1(y_1) = \int_{-\infty}^{\infty} f(y_1 - y_2)f(y_2)\, dy_2$$

is the p.d.f. of Y_1, and conclude that $g_1(y)$ is the p.d.f. of Y.

Solution: Using the change of variables formula, the p.d.f. of $X_i = W_i/2$, $i = 1, 2$, is

$$f_i(x) = h(2x) \cdot (2x)' = \frac{2}{\pi(1 + (2x)^2)} = \frac{2}{\pi(1+4x^2)},$$

since $w = 2x$ is the inverse function of $x = w/2$. Next, the Jacobian corresponding to the linear change of variables $Y_1 = X_1 + X_2$ and $Y_2 = X_1$ is

$$J = \begin{vmatrix} \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial y_2} \\ \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1,$$

where $v_1(y_1, y_2) = y_2$ and $v_2(y_1, y_2) = y_1 - y_2$. Since X_1 and X_2 are independent (remember that W_1, W_2 were), from the change of variables formula the joint p.d.f. of Y_1 and Y_2 is

$$g(y_1, y_2) = |-1| f(y_2, y_1 - y_2) = f_1(y_2)f_2(y_1 - y_2),$$

as desired. Writing $f = f_i$, for $i = 1, 2$, to get the marginal p.d.f. for Y_1 we integrate away the Y_2 variable,

$$g_1(y_1) = \int_{-\infty}^{\infty} g(y_1, y_2)\, dy_2 = \int_{-\infty}^{\infty} f(y_2)f(y_1 - y_2)\, dy_2.$$

$\square$

2. Let X_1, X_2 be independent random variables which are exponentially distributed with parameters λ_1, λ_2, respectively. Find the p.d.f. of $Y = X_1 + X_2$ using the convolution formula

$$f * g(x) = \int_a^b f(t)g(t-x)\,dt.$$

Hint: make the necessary modifications to your argument from Problem #1.

Solution: Let $f_i(x_i) = \lambda_i e^{-\lambda_i x_i}$ be the marginal p.d.f. of X_i, $i = 1, 2$. We have that the p.d.f. g of $Y = X_1 + X_2$ is given by the convolution formula $g = f_1 * f_2$,

$$\begin{aligned} g(y) &= f_1 * f_2(y) \\ &= \int_0^\infty f_1(t) f_2(t-y)\,dt \\ &= \int_0^\infty \lambda_1 e^{-\lambda_1 t} \cdot \lambda_2 e^{-\lambda_2 (t-y)}\,dt \\ &= \int_0^\infty \lambda_1 \lambda_2 e^{-(\lambda_1 t + \lambda_2 (t-y))}\,dt. \quad = \frac{\lambda_1 \lambda_2}{-\lambda_1 - \lambda_2} e^{-(\lambda_1 t + \lambda_2 (t-y))} \end{aligned}$$

□

3. Approximate $P(39.75 \le \bar{X} \le 41.25)$, where $\bar{X}$ is the mean of a random sample of size 28 from a distribution with mean $\mu = 40$ and variance $\sigma^2 = 4$.

Solution: We use the central limit theorem with $Z = (\bar{X} - \mu)/(\sigma/\sqrt{n})$,

$$\begin{aligned} P(39.75 \le \bar{X} \le 41.25) &= P\left(\frac{39.75 - 40}{(2/\sqrt{28})} \le Z \le \frac{41.25 - 40}{(2/\sqrt{28})}\right) \\ &= P(-.66 \le Z \le 3.31) \\ &\approx 1 - (1 - .7454) = \boxed{.7454}. \end{aligned}$$

□

4. A random sample of size $n = 18$ is taken from a distribution with p.d.f. $f(x) = 1 - .5x$, $0 \le x \le 2$. Approximate $P(.66 \le \bar{X} \le .83)$.

Solution: First note that the mean of the random variable X with p.d.f. $f(x)$ above is found in the usual way, $\mu = \int_0^2 xf(x)\,dx = \int_0^2 x(1-.5x)\,dx = \int_0^2 (x - .5x^2)\,dx = (1/2)x^2 - (1/6)x^3|_0^2 = 2 - 8/6 = 2/3$. Similarly, it is easy to see that $\sigma^2 = .222$. Setting $Z = (\bar{X} - \mu)/(\sigma/\sqrt{n})$, we have that Z is approximately standard normally distributed. So

$$\begin{aligned} P(.66 \le \bar{X} \le .83) &= P(-.01 \le Z \le 1.47) \\ &\approx .9292 - (1 - .5080) = \boxed{.4372}. \end{aligned}$$

□

5. Let X equal the weight of individual apple coming out of a particular barrel of apples. Suppose that $E(X) = 24$ and $\mathrm{Var}(X) = 2.2$. Let $\bar{X}$ be the sample mean of a random sample of $n = 20$ apples. Find $E(\bar{X})$, $\mathrm{Var}(\bar{X})$, and approximate $P(23 \le \bar{X} \le 24)$.

Solution: We have $E(\bar{X}) = 24$ and $\mathrm{Var}(\bar{X}) = \sum_{i=1}^{20} \frac{1}{n^2} 2.2 = 2.2/20 = .11$. Setting $Z = (\bar{X} - 24)/(\sqrt{.11})$, we have that Z is approximately standard normal, so $P(23 \le \bar{X} \le 24) = P\left(\frac{23-24}{\sqrt{.11}}) \le Z \le 0\right) = P(-.3015 \le Z \le 0) \approx .5 - (1 - .6179) = \boxed{.1179}$.

□