Math 1552, Integral Calculus

Section 8.4: Trigonometric Substitutions

Evaluate the following integrals using any method we have learned so far.

1. $\int \frac{x^2}{(x^2+4)^{3/2}}dx$

Trig Sub: Let $x = 2\tan\theta$, then $dx = 2\sec^2\theta d\theta$, and $x^2+4 = 4\sec^2\theta$. Then the integral becomes:

$$\int \frac{4\tan^2\theta}{(4\sec^2\theta)^{3/2}} \cdot 2\sec^2\theta d\theta = \int \frac{\tan^2\theta}{\sec\theta} d\theta.$$

Changing $\tan^2\theta$ to $\sec^2\theta - 1$ yields:

$$\int \frac{\sec^2\theta - 1}{\sec\theta} d\theta = \int (\sec\theta - \cos\theta) d\theta.$$

Evaluating the integral gives: $\ln|\sec\theta + \tan\theta| - \sin\theta + C$.

Using the fact that $\tan\theta = \frac{x}{2}$, we have that $\sin\theta = \frac{x}{\sqrt{x^2+4}}$ and $\sec\theta = \frac{\sqrt{x^2+4}}{2}$, so the final answer is:

$$\ln|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}| - \frac{x}{\sqrt{x^2+4}} + C.$$

2. $\int \frac{\sqrt{1-x^2}}{x^4}dx$

Trig Sub: Let $x = \sin\theta$, then $dx = \cos\theta d\theta$ and $\sqrt{1-x^2} = \cos\theta$, so the integral becomes:

$$\begin{aligned}
\int \frac{\cos\theta}{\sin^4\theta} \cdot \cos\theta d\theta &= \int \frac{\cos^2\theta}{\sin^4\theta} d\theta \\
&= \int \frac{1-\sin^2\theta}{\sin^4\theta} d\theta \\
&= \int (\csc^4\theta - \csc^2\theta) d\theta \\
&= \int [\csc^2\theta(1+\cot^2\theta) - \csc^2\theta] d\theta \\
&= \int \csc^2\theta \cot^2\theta d\theta \\
[u = \cot\theta] &= -\frac{1}{3}\cot^3\theta + C.
\end{aligned}$$

Since $x = \sin\theta$, $\cot\theta = \frac{\sqrt{1-x^2}}{x}$, so the final answer is:

$$-\frac{1}{3} \cdot \frac{(1-x^2)^{3/2}}{x^3} + C.$$

3. $\int \frac{x}{(4-x^2)^{3/2}} dx$

u-substitution: Let $u = 4 - x^2$, then $du = -2xdx$ and

$$-\frac{1}{2} \int \frac{1}{u^{3/2}} du = \frac{1}{\sqrt{4-x^2}} + C.$$

4. $\int \frac{dx}{e^x \sqrt{e^{2x}-9}}$

Trig sub: Let $e^x = 3\sec\theta$, then $e^x dx = 3\sec\theta\tan\theta d\theta$, or $dx = \frac{3\sec\theta\tan\theta}{3\sec\theta} d\theta = \tan\theta d\theta$.

Then the integral becomes:

$$\begin{aligned}
\int \frac{\tan\theta d\theta}{3\sec\theta\sqrt{9\sec^2\theta - 9}} &= \int \frac{\tan\theta}{9\sec\theta\tan\theta} d\theta \\
&= \frac{1}{9} \int \cos\theta d\theta \\
&= \frac{1}{9} \sin\theta + C \\
&= \frac{1}{9} \cdot \frac{\sqrt{e^{2x} - 9}}{e^x} + C \\
&= \frac{\sqrt{e^{2x} - 9}}{9e^x} + C.
\end{aligned}$$

5. $\int \sin^2(x)\cos^2(x)dx$

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)dx &= \int [\sin(x)\cos(x)]^2 dx \\
&= \int \left[\frac{1}{2}\sin(2x)\right]^2 dx \\
&= \frac{1}{4}\int \sin^2(2x)dx \\
&= \frac{1}{4}\int \frac{1}{2}[1-\cos(4x)]dx \\
&= \frac{1}{8}\left[x - \frac{1}{4}\sin(4x)\right] + C \\
&= \frac{x}{8} - \frac{1}{32}\sin(4x) + C.
\end{aligned}$$

6. $\int (x^2+1)e^{2x}dx$

Integration by parts, using the "shortcut" method:

u	dv
x^2+1	e^{2x}
$2x$	$\frac{1}{2}e^{2x}$
2	$\frac{1}{4}e^{2x}$
0	$\frac{1}{8}e^{2x}$

Then

$$\int (x^2+1)e^{2x}dx = \frac{1}{2}(x^2+1)e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C.$$