Math 1552, Integral Calculus
Section 8.7: Numerical Integration

Let θ be an angle in radians, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so that $\tan\theta = \frac{1}{2}$.

We can find the value of θ using the integral:

$$\theta = \tan^{-1}\left(\frac{1}{2}\right) = \int_0^{1/2} \frac{1}{1+x^2}dx.$$

1. Estimate the value of θ with the trapezoidal rule using $n = 4$ subintervals.
Solution: Here, $\Delta x = \frac{1}{8}$, so we will divide the interval into subintervals $[0, \frac{1}{8}]$, $[\frac{1}{8}, \frac{1}{4}]$, $[\frac{1}{4}, \frac{3}{8}]$, and $[\frac{3}{8}, \frac{1}{2}]$. Then applying the Trapezoidal rule using $f(x) = \frac{1}{1+x^2}$ gives:

$$T_4 = \frac{1}{2} \cdot \frac{1}{8}\left[f(0) + 2f(\frac{1}{8}) + 2f(\frac{1}{4}) + 2f(\frac{3}{8}) + f(\frac{1}{2})\right] \approx 0.46281.$$

2. Using the formula for error in the Trapezoidal rule, estimate the largest possible error for your answer to problem 1.
Solution: We first need to find the largest value for $|f''(c)|$ on the interval $[0, \frac{1}{2}]$. Note that for $f(x) = \frac{1}{1+x^2}$, the second derivative is $f''(x) = \frac{2(3x^2-1)}{(1+x^2)^3}$. The absolute value of this function is decreasing on the interval $[0, \frac{1}{2}]$ (taking the third derivative can check that formally), so we reach the maximum value when $x = 0$. Thus, we have:

$$|f''(c)| \leq |f''(0)| = 2.$$

Using the formula for error in the Trapezoidal Rule, we can now solve:

$$\begin{aligned} |E_4^T| &\leq \frac{|b-a|^3}{12n^2} \cdot \max|f''(c)| \\ &= \frac{(\frac{1}{2}-0)^3}{12 \cdot 4^2} \cdot 2 \\ &= \frac{1}{768} \approx 0.0013. \end{aligned}$$

3. The actual value is approximately 0.46365. What is the percent error in your estimate in problem 1?
Solution: Actual error is $0.46365 - 0.46281 = 0.00084$.
The percent error is:

$$\frac{0.46365 - 0.46281}{0.46365} \times 100 \approx 0.18\%.$$

4. Estimate the value of θ with Simpson's rule using $n = 6$ subintervals.
Solution: With $n = 6$, the endpoints are 0, $\frac{1}{12}$, $\frac{1}{6}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{5}{12}$, and $\frac{1}{2}$ and $\Delta x = \frac{1}{12}$. So:

$$S_6 = \frac{1}{3} \cdot \frac{1}{12} \left[f(0) + 4f(\frac{1}{12}) + 2f(\frac{1}{6}) + 4f(\frac{1}{4}) + 2f(\frac{1}{3}) + 4f(\frac{5}{12}) + f(\frac{1}{2}) \right] \approx 0.46365.$$

Note that we get almost the exact answer with this approximation.