Math 1552, Integral Calculus

Sections 10.2-10.3: Infinite Series, Integral Test

1. Use series to write the repeating decimal 0.31313131... as a rational number.

We can write:

$$\begin{aligned} 0.313131... &= 0.31 + 0.0031 + 0.000031 + 0.00000031 + ... \\ &= \frac{31}{100} + \frac{31}{10000} + \frac{31}{1000000} + ... \\ &= \frac{31}{100}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6} + ...\right) \\ &= \frac{31}{100}\sum_{k=0}^{\infty}\left(\frac{1}{100}\right)^k \\ &= \frac{31}{100} \cdot \frac{1}{1 - 1/100} \\ &= \frac{31}{100} \cdot \frac{99}{100} = \frac{31}{99}. \end{aligned}$$

2. Find the sum of each convergent series below, or explain why the series diverges.

$$\sum_{k=7}^{\infty} \frac{1}{(k-3)(k+1)}, \quad \sum_{k=0}^{\infty}(-1)^k, \quad \sum_{k=2}^{\infty} \frac{2^k + 1}{3^{k+1}}$$

(a) This series is telescoping. We can rewrite it as:

$$\begin{aligned} \sum_{k=7}^{\infty} \frac{1}{(k-3)(k+1)} &= \frac{1}{4}\sum_{k=7}^{\infty}\left[\frac{1}{k-3} - \frac{1}{k+1}\right] \\ &= \frac{1}{4}\left[\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right] \\ &\approx 0.1899 \end{aligned}$$

(b) Note the partial sums: $S_0 = 1$, $S_1 = 1 + (-1) = 0$, $S_2 = 1 + (-1) + 1 = 1$, etc, so the sequence of partial sums is:

$$\{S_n\} = 1, 0, 1, 0, 1, ...$$

This sequence does not have a limit at n approaches infinity, so the series **diverges**.

(c) This series is geometric. We need to reduce it until we can use the formula

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

Thus:

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{2^k+1}{3^{k+1}} &= \sum_{k=2}^{\infty} \frac{2^k+1}{3^k \cdot 3} \\
&= \frac{1}{3}\left[\sum_{k=2}^{\infty}\left(\frac{2}{3}\right)^k + \sum_{k=2}^{\infty}\left(\frac{1}{3}\right)^k\right] \\
&= \frac{1}{3}\left[\frac{(2/3)^2}{1-\frac{2}{3}} + \frac{(1/3)^2}{1-\frac{1}{3}}\right] = \frac{1}{2}.
\end{aligned}$$

3. Determine if each series below converges or diverges. JUSTIFY YOUR ANSWER FULLY using either the nth term divergence test or the integral test.

(a)

$$\sum_{k=1}^{\infty} \frac{e^k}{4+e^{2k}}$$

Solution: We will use the integral test and evaluate $\int_0^{\infty} \frac{e^x}{4+e^{2x}} dx$. Let $u = e^x$, then $du = e^x dx$. Then the integral becomes:

$$\begin{aligned}
\int_0^{\infty} \frac{e^x}{4+e^{2x}} dx &= \lim_{b\to\infty} \int_{x=0}^{x=b} \frac{du}{4+u^2} \\
&= \lim_{b\to\infty} \frac{1}{4} \int_1^{e^b} \frac{du}{1+\left(\frac{u}{2}\right)^2} \\
&= \lim_{b\to\infty} \frac{1}{2} \tan^{-1}\left(\frac{u}{2}\right)\Big|_1^{e^b} \\
&= \lim_{b\to\infty} \frac{1}{2}\left[\tan^{-1}\left(\frac{e^b}{2}\right) - \tan^{-1}\frac{1}{2}\right] \\
&\approx \frac{1}{2}\left(\frac{\pi}{2} - 0.4636\right) \approx 0.5536.
\end{aligned}$$

Thus, the integral converges, and so does the series.

(b)

$$\sum_{k=1}^{\infty} \frac{5k^2+8}{7k^2+6k+1}$$

Solution: Since

$$\lim_{k\to\infty} \frac{5k^2+8}{7k^2+6k+1} = \frac{5}{7} \neq 0,$$

the series DIVERGES by the nth term test.