Math 1552, Integral Calculus

Sections 10.3-10.5: Convergence Tests

Determine whether the following series converge or diverge. Justify your answers using the tests we discussed in class.

(1)

$$\sum_{k=1}^{\infty} \frac{3^{2k}}{8^k - 3}$$

Solution: Note that $8^k - 3 < 8^k$, and thus $\frac{1}{8^k-3} > \frac{1}{8^k}$, so we have:

$$\frac{3^{2k}}{8^k - 3} > \frac{3^{2k}}{8^k} = \left(\frac{9}{8}\right)^k.$$

Since the series $\sum \left(\frac{9}{8}\right)^k$ is a geometric series with $r = \frac{9}{8} > 1$, it DIVERGES. Since our original series is bigger, it also DIVERGES by the Basic Comparison Test.

(2)

$$\sum_{k=1}^{\infty} \frac{k+2}{\sqrt{k^5+4}}$$

Solution: Note that $k^5 + 4 > k^5$, and thus $\sqrt{k^5+4} > \sqrt{k^5} = k^{5/2}$. Then:

$$\frac{k+2}{\sqrt{k^5+4}} < \frac{k+2}{k^{5/2}}.$$

Also note that for $k \geq 2$, $k + 2 \leq 2k$, so we can write:

$$\frac{k+2}{\sqrt{k^5+4}} < \frac{2k}{k^{5/2}} = 2 \cdot \frac{1}{k^{3/2}}.$$

Since the series $\sum_{k=2}^{\infty} \frac{1}{k^{3/2}}$ is a p-series with $p > 1$, it converges, and thus $2 \cdot \sum_{k=2}^{\infty} \frac{1}{k^{3/2}}$ also converges. Our original series is smaller, so we see by the Basic Comparison Test that $\sum_{k=2}^{\infty} \frac{k+2}{\sqrt{k^5+4}}$ also converges. We can always add a finite number of terms to a convergent series to obtain another convergent series, and thus the series $\sum_{k=1}^{\infty} \frac{k+2}{\sqrt{k^5+4}}$ must also converge.

(3)

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$$

Solution: Since $\frac{d}{dx}[\ln x] = \frac{1}{x}$, the integral test looks like a good tool here. As $k \geq 2$, $\ln k > 0$, so the function is continuous and positive. Let's check that it is also decreasing. Set $f(x) = \frac{1}{x(\ln x)^3}$. Then:

$$f'(x) = \frac{-(\ln x)^3 - 3(\ln x)^2}{x^2(\ln x)^6}.$$

Note that since we are taking $x \geq 2$ and $\ln x > 0$, $f'(x) < 0$. Now we can apply the integral test. Letting $u = \ln x$ and $du = \frac{1}{x}$, we have:

$$\begin{aligned}
\int_2^\infty \frac{dx}{x(\ln x)^3} &= \lim_{N\to\infty} \int_{\ln 2}^{\ln N} \frac{du}{u^3} \\
&= \lim_{N\to\infty} -\frac{1}{2u^2}\Big|_{\ln 2}^{\ln N} \\
&= \lim_{N\to\infty} -\frac{1}{2}\left[\frac{1}{(\ln N)^2} - \frac{1}{(\ln 2)^2}\right] \\
&= \frac{1}{2(\ln 2)^2},
\end{aligned}$$

which is a finite number. Thus the integral and the series both converge.