Math 1552, Integral Calculus

Section 10.4: Comparison and Limit Comparison Tests

Determine whether the following series converge or diverge. Justify your answers using any of the tests we discussed in class.

(1)

$$\sum_{k=1}^{\infty} k \tan\left(\frac{1}{k}\right)$$

Using the divergence test, note that:

$$\begin{aligned}
\lim_{n\to\infty} n\tan\left(\frac{1}{n}\right) &= \lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}} \\
&\equiv \lim_{n\to\infty} \frac{-\frac{1}{n^2}\sec^2\left(\frac{1}{n}\right)}{-\frac{1}{n^2}} \\
&= \sec^2(0) = 1 \neq 0,
\end{aligned}$$

so the series **diverges**.

(2)

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + ...$$

First, let's rewrite the series in summation notation:

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + .. = \sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)}.$$

We will use the limit comparison test with the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, which converges since it is a p-series with $p = 2 > 1$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2}{(2n-1)(2n+1)} \\
&= \lim_{n\to\infty} \frac{n^2}{4n^2-1} \\
&= \frac{1}{4}.
\end{aligned}$$

Since $0 < \frac{1}{4} < \infty$, the series $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + ...$ must also **converge**.

(3)

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$$

Using the Limit Comparison Test, comparing to $\sum \frac{1}{n^2}$, which converges (p-series with $p = 2 > 1$):

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{1}{n\sqrt{n^2-1}} \cdot \frac{n^2}{1} \\
&= \lim_{n\to\infty} \frac{\sqrt{n^4}}{\sqrt{n^2}\sqrt{n^2-1}} \\
&= \lim_{n\to\infty} \sqrt{\frac{n^4}{n^4-n^2}} \\
&= \sqrt{\lim_{n\to\infty} \frac{n^4}{n^4-n^2}} \\
&= \sqrt{1} = 1.
\end{aligned}$$

Since $0 < 1 < \infty$, both series must **converge**.

(4)

$$\sum_{k=1}^{\infty} \frac{\ln k}{k^4}$$

Solution: Since $\ln k < k$ when $k > 1$, we can see that:

$$\frac{\ln k}{k^4} < \frac{k}{k^4} = \frac{1}{k^3}.$$

As $\sum \frac{1}{k^3}$ is a convergent p-series, the series will *converge* by the basic comparison test. Alternately, one could also show convergence using the integral test.