Math 1552, Integral Calculus

Sections 10.5: Ratio and Root Tests

Determine whether the following series converge or diverge. Justify your answers using any of the tests we discussed in class.

(1)

$$\sum_{k=1}^{\infty} \frac{(2k)^k}{k!}$$

We will apply the ratio test:

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{(2(n+1))^{n+1}}{(n+1)!} \cdot \frac{n!}{(2n)^n} \\
&= \lim_{n\to\infty} \frac{2^{n+1} \cdot (n+1)^{n+1}}{(n+1)n!} \cdot \frac{n!}{2^n n^n} \\
&= \lim_{n\to\infty} 2\left(\frac{n+1}{n}\right)^n \\
&= \lim_{n\to\infty} 2\left(1+\frac{1}{n}\right)^n \\
&= 2e > 1,
\end{aligned}$$

so the series **diverges**.

(2)

$$\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{2k^2}$$

Apply the root test:

$$\begin{aligned}
R &= \lim_{n\to\infty} (a_n)^{1/n} \\
&= \lim_{n\to\infty} \left[\left(\frac{n}{n+1}\right)^{2n^2}\right]^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{2n} \\
&= \lim_{n\to\infty} \left[\frac{1}{\left(\frac{n+1}{n}\right)^n}\right]^2
\end{aligned}$$

$$= \left[\frac{1}{\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n} \right]^2$$

$$= \frac{1}{e^2} < 1,$$

so the series **converges**.

(3)

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{4^n 2^n n!}$$

Using the ratio test, we have:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1) \cdot (2(n+1)-1)}{4^{n+1} 2^{n+1} (n+1)!} \cdot \frac{4^n 2^n n!}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)} \\
&= \lim_{n\to\infty} \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1) \cdot (2n+1)}{4^n \cdot 4 \cdot 2^n \cdot 2 \cdot (n+1) n!} \cdot \frac{4^n 2^n n!}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)} \\
&= \lim_{n\to\infty} \frac{2n+1}{8(n+1)} \\
&= \frac{1}{4} < 1,
\end{aligned}$$

so the series **converges** by the ratio test.

(4) Suppose $r > 0$. Find the values of r, if any, for which $\sum_{k=1}^{\infty} \frac{r^k}{k^r}$ converges.

Applying the ratio test, we have:

$$\begin{aligned}
\lim_{n\to\infty} \frac{r^{n+1}}{(n+1)^r} \cdot \frac{n^r}{r^n} &= \lim_{n\to\infty} r \cdot \left(\frac{n}{n+1} \right)^r \\
&= r \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^r \\
&= r \left(\lim_{n\to\infty} \frac{n}{n+1} \right)^r \\
&= r \cdot 1^r = r,
\end{aligned}$$

so the series converges when $0 < r < 1$.