Math 1552, Integral Calculus
Sections 5.2-5.3: The Definite Integral

1. (*Applying the Definite Integral*) A marketing company is trying a new campaign. The campaign lasts for three weeks, and during this time, the company finds that it gains customers as a function of time according to the formula:

$$C(t) = 5t - t^2,$$

where t is time in weeks and the number of customers is given in thousands.
Using the general form of the definite integral,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*),$$

calculate the **average** number of customers gained during the three-week campaign.
Solution: First, note that since average value is defined as $AV = \frac{1}{b-a}\int_a^b f(x)dx$, we can use the Riemann sum formula to obtain (the term $b-a$ will cancel):

$$AV = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} C(x_i^*).$$

In this problem, $a = 0$ and $b = 3$. Breaking the interval into n equal pieces would give $\Delta x = \frac{3}{n}$. To find each right-hand endpoint, we can set:

$$x_i^* = a + i\Delta x = 0 + \frac{3i}{n} = \frac{3i}{n},$$

and thus

$$C(x_i^*) = 5\left(\frac{3i}{n}\right) - \left(\frac{3i}{n}\right)^2 = \frac{15}{n}i - \frac{9}{n^2}i^2.$$

Now plugging into the summation:

$$\begin{aligned}
\sum_{i=1}^{n} C(x_i^*) &= \sum_{i=1}^{n}\left(\frac{15}{n}i - \frac{9}{n^2}i^2\right) \\
&= \frac{15}{n}\sum_{i=1}^{n} i - \frac{9}{n^2}\sum_{i=1}^{n} i^2 \\
&= \frac{15}{n}\cdot\frac{n(n+1)}{2} - \frac{9}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} \\
&= \frac{15(n+1)}{2} - \frac{9(n+1)(2n+1)}{6n}.
\end{aligned}$$

Using this expression, we can now find average value:

$$
\begin{aligned}
AV &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} C(x_i^*) \\
&= \lim_{n\to\infty} \frac{1}{n}\left(\frac{15(n+1)}{2} - \frac{9(n+1)(2n+1)}{6n}\right) \\
&= \lim_{n\to\infty} \left(\frac{15(n+1)}{2n} - \frac{9(n+1)(2n+1)}{6n^2}\right) \\
&= \frac{15}{2} - \frac{18}{6} \\
&= 4.5,
\end{aligned}
$$

so the company gained an average of 4,500 customers weekly during the campaign.

2. Explain why the following property is true:

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

Can you find an example where the inequality is strict?

Solution: Note that $|\int_a^b f(x)dx|$ represents "absolute net area," while $\int_a^b |f(x)|dx$ is "total area." In general, $total \quad area \geq |net \quad area|$.

For an example where the inequality is strict, let $f(x) = x$ on the interval $[-1, 2]$. If you draw a picture, we can find the area geometrically (all triangles). The area of the triangle from 0 to 2 is 2, and the area of the triangle from -1 to 0 is $-\frac{1}{2}$ (dips below the x-axis), so:

$$\left|\int_{-1}^2 x dx\right| = \frac{3}{2}, \quad \int_{-1}^2 |x| dx = \frac{5}{2}.$$

3. Evaluate $\int_0^2 |x-1|dx$ using integral properties from class (you may use geometry, or a Riemann Sum).

Solution: Draw a picture. Note that $\int_0^1 |x-1|dx = \int_1^2 |x-1|dx$, so $\int_0^2 |x-1|dx = 2\int_0^1 |x-1|dx$.

Using geometry, we can draw the triangle from 0 to 1 to see the area is $\frac{1}{2}\cdot 1 \cdot 1 = \frac{1}{2}$, so: $\int_0^2 |x-1|dx = 2\cdot\frac{1}{2} = 1$.

Alternately, using Riemann Sums, we will break the interval $[0, 1]$ into n subintervals of length $\Delta x = \frac{1}{n}$, and endpoints $x_i = \frac{i}{n}$. Taking $x_i^* = \frac{i}{n}$ as the right-hand endpoint of each interval, $1 \leq i \leq n$, we see the area of each rectangle is $f(x_i^*)\Delta x = \left|\frac{i}{n} - 1\right|\frac{1}{n}$. As $1 \geq \frac{i}{n}$, $\left|\frac{i}{n} - 1\right| = 1 - \frac{i}{n}$, and thus:

$$\begin{aligned}
\int_0^2 |x-1|dx &= 2\int_0^1 |x-1|dx \\
&= 2\lim_{n\to\infty}\sum_{i=1}^n \left(1 - \frac{i}{n}\right)\frac{1}{n} \\
&= 2\lim_{n\to\infty}\frac{1}{n}\left(\sum_{i=1}^n 1 - \frac{1}{n}\sum_{i=1}^n i\right) \\
&= 2\lim_{n\to\infty}\frac{1}{n}\left(n - \frac{1}{n}\cdot\frac{n(n+1)}{2}\right) \\
&= 2\lim_{n\to\infty}\left(1 - \frac{n+1}{2n}\right) \\
&= 2\cdot\left(1 - \frac{1}{2}\right) = 1.
\end{aligned}$$