Math 1552, Integral Calculus

Sections 7.2: Separable Differential Equations

1. Solve the initial value problem:

$$y' = x\sqrt{\frac{1-y^2}{1-x^2}}, \quad y(0) = 0.$$

Solution: Separate the variables, then integrate:

$$\int \frac{1}{\sqrt{1-y^2}} dy = \int \frac{x}{\sqrt{1-x^2}} dx,$$

so

$$\sin^{-1}(y) = -\sqrt{1-x^2} + C$$

and thus:

$$y = \sin(-\sqrt{1-x^2} + C).$$

Setting $y(0) = 0$ yields: $\sin(-1 + C) = 0$, so $-1 + C = 0$ and $C = 1$. Then the particular solution is:

$$y = \sin(-\sqrt{1-x^2} + 1).$$

2. Twenty percent of the candy in your Halloween bucket disappears after one hour. If the candy is disappearing exponentially, determine:

(a) the percent of candy after three hours, and

(b) the amount of time it will take until you have less than one percent of your candy left.

(a) After one hour, 80% is left, so $A(1) = 0.8A_0$. Then since $A(t) = A_0e^{rt}$, we can solve to find $e^r = 0.8$. Thus, $A(t) = A_0(0.8)^t$. So:

$$A(3) = A_0(0.8)^3 = 0.512A_0,$$

so 51.2% will remain.

(b) We want to find t so that $0.01A_0 = A_0(0.8)^t$. Solving for t, we have:

$$t = \frac{\ln(0.01)}{\ln(0.8)} \approx 20.64 \quad hours.$$

3. As you go trick-or-treating, you walk with a velocity of $v(t) = \frac{e^t}{1+e^t}$ feet per minute. How far have you traveled after $\ln 5$ minutes?

We want to find:

$$\begin{aligned}\int_0^{\ln 5} \frac{e^t}{1+e^t} dt &= \ln(1+e^t)|_0^{\ln 5}\\ &= \ln(1+e^{\ln 5}) - \ln(1+e^0)\\ &= \ln 6 - \ln 2\\ &= \ln 3 \quad feet.\end{aligned}$$

4. Review: evaluate each integral.

$$\int \frac{\log_3 x^4}{x} dx$$

$$\begin{aligned}&= \int \frac{4\log_3 x}{x} dx\\ &= \frac{4}{\ln 3}\int \frac{\ln x}{x} dx\\ &= \frac{4}{\ln 3}(\ln x)^2 + C.\end{aligned}$$

$$\int (1+\ln x)\cot(x\ln x)dx$$

Let $u = x\ln x$, then $du = (\ln x + 1)dx$.

$$\begin{aligned}&= \int \cot u du\\ &= \ln|\sin(x\ln x)| + C.\end{aligned}$$