Math 1552, Integral Calculus

Test 3 Review, Sections 10.1-10.7

Many of the following problems come from the recitation worksheets. These problems are denoted with an "(R)" next to the problem number.

1. **Terminology review**: complete the following statements.

(a) A geometric series has the general form ____________.
The series converges when ____________ and diverges when ____________.

(b) A p-series has the general form ____________. The series converges when ____________ and diverges when ____________. To show these results, we can use the ____________ test.

(c) The harmonic series ____________ and telescoping series ____________.

(d) If you want to show a series converges, compare it to a ____________ series that also converges. If you want to show a series diverges, compare it to a ____________ series that also diverges.

(e) If the direct comparison test does not have the correct inequality, you can instead use the ____________ ____________ test. In this test, if the limit is a ____________ number (not equal to ____________), then both series converge or both series diverge.

(f) In the ratio and root tests, the series will ____________ if the limit is less than 1 and ____________ if the limit is greater than 1. If the limit equals 1, then the test is ____________.

(g) If $\lim_{n\to\infty} a_n = 0$, then what do we know about the series $\sum_n a_n$? ____________

(h) A power series has the general form: ____________. To find the radius of convergence R, use either the ____________ or ____________ test. The series converges ____________ when $|x - c| < R$. To find the interval of convergence, don't forget to check the ____________.

2. Let $\{a_n\}$ and $\{b_n\}$ be a sequences of non-negative terms. Are the following statements *always* true or sometimes false?

(a) If $\lim_{n\to\infty} a_n = L$, then the series $\sum_n a_n = L$.
(b) If $\{a_n\}$ is decreasing and bounded, then $\lim_{n\to\infty} a_n = g.l.b.$.
(c) If $\lim_{n\to\infty} a_n = 0$, then $\{a_n\}$ converges to 0.
(d) If $\lim_{n\to\infty} a_n = 0$, then $\sum_n a_n$ converges.

(e) If $\sum_n a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(f) If $\sum_n a_n$ diverges, then $\lim_{n\to\infty} a_n \neq 0$.

(g) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_n a_n$ diverges.

(h) If $\lim_{n\to\infty} a_n \neq 0$, then $\{a_n\}$ diverges.

(i) If $\int_1^\infty f(x)dx = L$, where $0 < L < \infty$, then $\sum_n f(n) = L$.

(j) If $\sum_n b_n$ converges and $a_n > b_n$ for all $n \geq 1$, then $\sum_n a_n$ also converges.

(k) If $\sum_n b_n$ diverges and $a_n > b_n$ for all $n \geq 1$, then $\sum_n a_n$ also diverges.

(l) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_n b_n$ diverges, then $\sum_n a_n$ also diverges.

3 (R). For each sequence below, find the l.u.b. and g.l.b., and determine if the sequence is monotonic.

$$\{\sin(n\pi)\}, \quad \left\{(-1)^{n+1}\frac{1}{5^n}\right\}, \quad \left\{\frac{n+1}{n}\right\}$$

4 (R). Determine whether or not each sequence converges. If so, find the limit.

(a) $\left\{\frac{2n^2}{\sqrt{9n^4+1}}\right\}$

(b) $\left\{\left(1-\frac{1}{8n}\right)^n\right\}$

(c) $\left\{\frac{n!}{e^n}\right\}$

(d) $\left\{\left(\frac{n}{n+5}\right)^n\right\}$

5 (R). Use series to write the repeating decimal 0.31313131... as a rational number.

6 (R). **Find the sum** of each convergent series below, or explain why the series diverges.

(a) $\sum_{k=7}^\infty \frac{1}{(k-3)(k+1)}$

(b) $\sum_{k=0}^\infty (-1)^k$

(c) $\sum_{k=2}^\infty \frac{2^k+1}{3^{k+1}}$

7 (R). Determine if each series below converges or diverges. JUSTIFY YOUR ANSWER FULLY using any of the convergence tests from class.

(a) $\sum_{k=1}^\infty \frac{e^k}{4+e^{2k}}$

(b) $\sum_{k=1}^\infty \frac{5k^2+8}{7k^2+6k+1}$

(c) $\sum_{k=1}^\infty \frac{3^{2k}}{8^k-3}$

(d) $\sum_{k=1}^\infty \frac{k+2}{\sqrt{k^5+4}}$

(e) $\sum_{k=2}^\infty \frac{1}{k(\ln k)^3}$

(f) $\sum_{k=1}^{\infty} k \tan\left(\frac{1}{k}\right)$

(g) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + ...$

(h) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

(i) $\sum_{k=1}^{\infty} \frac{\ln k}{k^4}$

(j) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n-1)}{4^n 2^n n!}$

8. Determine if each series below converges or diverges. JUSTIFY YOUR ANSWER FULLY using any of the convergence tests from class.

(a) $\sum_{k=1}^{\infty} \frac{(2k)^k}{k!}$

(b) $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{2k^2}$

(c) $\sum_{n=2}^{\infty} \frac{2-\cos(n)}{n^2-1}$

(d) $\sum_{n=1}^{\infty} \frac{1+\sin(e^n)}{n^{3/2}}$

(e) $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

9 (R). Suppose $r > 0$. Find the values of r, if any, for which $\sum_{k=1}^{\infty} \frac{r^k}{k^r}$ converges.

10 (R). Determine whether the following alternating series converge absolutely, converge conditionally, or diverge. Justify your answers using the tests we discussed in class.

(a) $\sum_{k=2}^{\infty}(-1)^{k+1}\frac{3k}{\sqrt{k^3+4}}$

(b) $\sum_{k=2}^{\infty}(-1)^{k}\frac{k}{k^4-1}$

(c) $\sum_{k=0}^{\infty}(-1)^{k}\frac{k}{5^k+2^k}$

(d) $\sum_{k=3}^{\infty}(-1)^{k}\frac{1}{k\ln k\sqrt{\ln\ln k}}$

11. Find the radius and interval of convergence of the following power series:

(a) $\sum_{k=2}^{\infty}\left(\frac{k}{k-1}\right)\frac{(x+2)^k}{2^k}$

(b) $\sum_{k=1}^{\infty}\frac{k}{3^k(k^2+1)}(x-5)^k$

(c) $\sum_{k=1}^{\infty}\frac{5^k}{\sqrt{k}}(3-2x)^k$

(d) $\sum_{n=0}^{\infty}\frac{3^{2n-1}x^n}{4^n}$

(e) $\sum_{n=1}^{\infty}\frac{(3x+2)^n}{\sqrt{n}}$

12. Do the following series converge absolutely, converge conditionally, or diverge? Justify your answers using the convergence tests from class.

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{n!}{n^n}$

(b) $\sum_{n=1}^{\infty}(-1)^{3n+1} \frac{(1+5n^4)^n}{(1+3n^2)^{2n}}$

(c) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$

(d) $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n}{(n+1)^2}$

13. For what values of p does the series $\sum_{n=1}^{\infty} n^{\frac{1}{n-p}}$ converge? For what values does it diverge?

14. Consider the power series $f(x) = \sum_{n=0}^{\infty} \left(\frac{x^2+2}{6}\right)^n$.

(a) Find the interval of convergence for this series.

(b) For all values of x that lie in the interval of convergence, what is the sum of this series as a function of x?

15. (a) Give an example of a divergent monotonic sequence.

(b) Give an example of a divergent bounded sequence.

16. Let $a_n = \ln\left(\frac{n+1}{n}\right)$.

(a) Does the sequence $\{a_n\}$ converge or diverge? If it converges, find the limit.

(b) Find a formula for the n-th partial sum $s_n = a_1 + a_2 + \cdots + a_n$. Simplify.

(c) Does the series $\sum_{k=1}^{\infty} a_k$ converge or diverge? If it converges, find the sum.

17. (a) Does the series $\sum_{n=2}^{\infty} \frac{(\pi-1)^{n+1}}{e^{n-1}}$ converge or diverge? If it converges, find the sum.

(b) Does the series $\sum_{n=1}^{\infty} 3^{-\frac{1}{n}}$ converge or diverge?

Answers

2. Statements (b), (c), (e), (g), and (k) are true.

3. (a) l.u.b.=g.l.b.=0, monotonic; (b) l.u.b.=$\frac{1}{5}$, g.l.b.=$-\frac{1}{25}$, not monotonic; (c) l.u.b.=2, g.l.b.=1, monotonic

4. (a) converges to $\frac{2}{3}$, (b) converges to $e^{-1/8}$, (c) diverges, (d) converges to $\frac{1}{e^5}$

5. $\frac{31}{99}$

6. (a) $\frac{319}{1{,}680}$, (b) diverges, (c) $\frac{1}{2}$

7. (a) converges by integral test, (b) diverges by divergence test, (c) diverges by direct comparison, (d) converges by direct or limit comparison, (e) converges by integral test, (f) diverges by divergence test, (g) converges by direct comparison, (h) converges by limit comparison, (i) converges by direct comparison, (j) converges by ratio test

8. (a) diverges by ratio test, (b) converges by root test, (c) converges by direct comparison, (d) converges by direct comparison, (e) converges by integral test

9. converges when $0 < r < 1$

10. (a) converges conditionally (limit comparison and alternating series test), (b) converges absolutely (limit comparison), (c) converges absolutely (ratio test), (d) converges conditionally (integral test and alternating series test), (e) conditionally convergent (direct comparison and alternating series test), (f) diverges (nth term test), (g) conditionally convergent (direct comparison and alternating series test), (h) absolutely convergent (geometric), (i) conditionally convergent (integral test and alternating series test), (j) absolutely convergent (limit comparison test)

11. (a) $R = 2$, $I.C. = (-4, 0)$, (b) $R = 3$, $I.C. = [2, 8)$ (c) $R = \frac{1}{10}$, $I.C. = \left(\frac{7}{5}, \frac{8}{5}\right]$, (d) $R = \frac{4}{9}$, $I.C. = \left(-\frac{4}{9}, \frac{4}{9}\right)$, (e) $R = \frac{1}{3}$, $I.C. = \left[-1, -\frac{1}{3}\right)$

12. (a) converges absolutely (use ratio test); (b) converges absolutely (use root test) ; (c) converges absolutely (use integral test); (d) converges conditionally (use limit comparison)

13. always diverges (use nth term test)

14. $I.C. = (-2, 2)$, sum is $\frac{6}{4-x^2}$

15. answers may vary

16. (a) converges to 0; (b) $s_n = \log(n+1)$; (c) diverges

17. (a) converges to $\frac{(\pi-1)^3}{e-\pi+1}$; (b) diverges by the n^{th} term test