

Math 1552 1:55 Version B  
Spring 2019  
Midterm 3  
April 12, 2019  
Time Limit: 50 Minutes

Name (Print): \_\_\_\_\_  
Canvas email: \_\_\_\_\_  
TA: \_\_\_\_\_

GT ID:

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Sign Your Name: \_\_\_\_\_

This exam contains 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **This exam is double-sided.**
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Please circle or box in your final answer.
- If you need extra space, you may use the back side of this cover page.

Problem	Points	Score
1	34	
2	15	
3	10	
4	25	
5	16	
Total:	100	

### **SCRATCH WORK**

If you want work on this page to be graded, please make a note on the corresponding problem.

1. Consider the alternating series

$$\sum_{k=2}^{\infty} (-1)^k \frac{1}{k \ln k}.$$

- (a) (16 points) Does the series converge absolutely, converge conditionally, or diverge? Justify your answer fully using the convergence tests from class.

**Grading Rubric:**

- (1 point for absolute value) We will first consider  $\sum_k |a_k| = \sum_{k=2}^{\infty} \frac{1}{k \ln k}$ .
- (3 points) Note that the terms are decreasing since  $a_n = \frac{1}{n \ln n} > \frac{1}{(n+1) \ln(n+1)} = a_{n+1}$ . If we let  $f(x) = \frac{1}{x \ln x}$ , this function is positive, continuous, and decreasing, so we can use the:
- (2 points to state test) Integral Test:
- (4 points to apply the Integral Test):

$$\begin{aligned} \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x \ln x} dx &= \lim_{b \rightarrow \infty} \ln |\ln x| \Big|_2^b \\ &= \lim_{b \rightarrow \infty} [\ln |\ln b| - \ln |\ln 2|] = \infty. \end{aligned}$$

- (2 points) The integral diverges, so the series does NOT converge absolutely.
  - (2 points) We already stated above that the terms are decreasing, and we also see that  $\lim_{k \rightarrow \infty} \frac{1}{k \ln k} = 0$ .
  - (2 points for answer matching above work) Thus,  $\sum_{k=2}^{\infty} (-1)^k \frac{1}{k \ln k}$  **converges conditionally**.
- (b) (18 points) Find the radius and interval of convergence of the power series below. HINT: Your answer to (a) should help.

$$\sum_{k=2}^{\infty} \frac{4^k}{k \ln k} (x+2)^k.$$

**Grading Rubric:**

- (2 points) Put either the coefficient term, or everything, into the Ratio Test.

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{4^{n+1}}{(n+1) \ln(n+1)} \cdot \frac{n \ln(n)}{4^n}.$$

- (5 points) Evaluate the limit.

$$L = \lim_{n \rightarrow \infty} 4 \frac{n+1}{n} \frac{\ln n}{\ln(n+1)} = 4 \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)} = 4 \cdot 1 \cdot \lim_{n \rightarrow \infty} \frac{1/n}{1/(n+1)} = 4 \cdot 1 \cdot 1 = 4.$$

- (2 points) Find  $R$ :

$$R = \frac{1}{L} = \frac{1}{4}.$$

- (3 points) Find an open interval on which the series converges absolutely. The series converges when  $|x+2| < \frac{1}{4} \iff -\frac{9}{4} < x < -\frac{7}{4}$ .
- (2 points) Determine convergence or diverges at  $x = -\frac{7}{4}$ : we obtain the series  $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ . From part (a), this series diverges.
- (2 points) Determine convergence or diverges at  $x = -\frac{9}{4}$ : we obtain the series  $\sum_{k=2}^{\infty} \frac{(-1)^n}{k \ln k}$ . From part (a), this series converges conditionally.
- (2 point) Summarize into the overall interval of convergence: the series converges on the closed interval  $[-\frac{9}{4}, -\frac{7}{4})$ .

2. (15 points) Find a MacLaurin series for the function below. Write your answer in sigma notation that is simplified as far as possible.

$$g(x) = \int_0^x \frac{\cos(t^3)}{t} dt$$

**Grading Rubric:**

- (2 points) Start with:

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

- (3 points) Replace  $x$  with  $t^3$ :

$$\cos(t^3) = \sum_{k=0}^{\infty} (-1)^k \frac{(t^3)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{6k}}{(2k)!}$$

- (2 points) Divide by  $t$ :

$$\frac{\cos(t^3)}{t} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{6k-1}}{(2k)!}$$

- (5 points) Integrate:

$$\int_0^x \frac{\cos(t^3)}{t} dt = \int_0^x \left( \sum_{k=0}^{\infty} (-1)^k \frac{t^{6k-1}}{(2k)!} \right) dt = \left[ \sum_{k=0}^{\infty} (-1)^k \frac{t^{6k}}{(2k)!(6k)} \right]_0^x$$

- (3 points) Replace  $t$  with  $x$ :

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{6k}}{(2k)!(6k)}.$$

3. **True or False?** Determine if each statement below is always true or sometimes false. FILL IN THE CIRCLE COMPLETELY for each statement.

- (a) **(2 points) FALSE** If  $\int_1^{\infty} f(x)dx = L$ , where  $0 < L < \infty$ , then  $\sum_n f(n) = L$ .
- (b) **(2 points) TRUE** If  $\sum_k a_k$  is an alternating series and  $\sum_k |a_k|$  diverges, then  $\sum_k a_k$  cannot converge absolutely.
- (c) **(2 points) FALSE** For any Taylor polynomial, the error in the approximation is no more than the magnitude of the  $(n+1)^{st}$  term.
- (d) **(2 points) TRUE** If  $\sum_n a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .
- (e) **(2 points) FALSE** If  $\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} < 1$ , then the series  $\sum_k a_k$  converges.

4. (a) (15 points) Estimate  $\sqrt[3]{1.2}$  using a 2nd degree Taylor polynomial of  $f(x) = \sqrt[3]{x}$  centered at  $a = 1$ . You do not need to fully simplify your final answer.

**Grading Rubric:**

- Set  $f(x) = \sqrt[3]{x} = x^{1/3}$ .
- (4 points) Calculate  $f'(x) = \frac{1}{3}x^{-2/3}$  and  $f''(x) = -\frac{2}{9}x^{-5/3}$ .
- (3 points) Find  $f(1) = 1$ ,  $f'(1) = \frac{1}{3}$ , and  $f''(1) = -\frac{2}{9}$ .
- (6 points) Find  $P_2(x)$ :

$$P_2(x) = 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2.$$

- (2 points) Plug in  $x = 1.2$ . Answer does not have to be simplified.

$$P_2(1.1) = 1 + \frac{1}{3} \cdot \frac{1}{5} - \frac{1}{9} \cdot \frac{1}{25} = 1 + \frac{1}{15} - \frac{1}{225}.$$

- (b) (10 points) Find the maximum error in your estimate from part (a). You do not need to fully simplify your final answer. Recall the Taylor remainder formula:  $|R_n(x)| \leq \max |f^{(n+1)}(c)| \cdot \frac{|x-a|^{n+1}}{(n+1)!}$ .

**Grading Rubric:**

- (2 points) Since  $n = 2$ , find  $f^{(3)}(x) = \frac{10}{27}x^{-8/3}$ .
- (3 points) On the interval  $[1, 1.2]$ , this function is decreasing, so  $M = \max |f^{(3)}(x)| = f^{(3)}(1) = \frac{10}{27}$ .
- (4 points) Plug everything into the formula:

$$|R_2(1.1)| \leq \frac{10}{27} \cdot \frac{|1.2-1|^3}{3!}$$

- (1 point) Simplify a little, but do not need to simplify fully.

$$= \frac{10}{27} \cdot \frac{1}{125} \cdot \frac{1}{6} = \frac{1}{2,025}.$$

5. (16 points) Determine if the series below converges or diverges. Justify your answer using the convergence tests from class. Be sure that you (1) name the test and state the conditions needed for the test you are using, (2) show work for the test that requires some math, and (3) state a conclusion that explains why the test shows convergence or divergence.

$$\sum_{k=1}^{\infty} \frac{k^3}{6^k + 2^k}$$

- (2 points) State the test and conditions. We can start with the Direct (or Basic) Comparison Test.
- (3 points) We can compare this series to the series  $\sum_k \frac{k^3}{6^k}$ .
- (5 points) By the Ratio Test:

$$\lim_{n \rightarrow \infty} \frac{(n+1)^3}{6^{n+1}} \cdot \frac{6^n}{n^3} = \lim_{n \rightarrow \infty} \frac{1}{6} \cdot \left( \frac{n+1}{n} \right)^3 = \frac{1}{6} \cdot 1 = \frac{1}{6} < 1,$$

so  $\sum_k \frac{k^3}{6^k}$  converges.

- (4 points) Conduct the Direct Comparison Test. We note that  $6^k + 2^k > 6^k$  for all  $k \geq 1$ . Thus:

$$\frac{k^3}{6^k + 2^k} < \frac{k^3}{6^k}.$$

We have shown every term in the given series is smaller than the corresponding term in the comparison series.

- (2 points) Write out a conclusion. Since we have found a bigger series that converges, the series  $\sum_{k=1}^{\infty} \frac{k^3}{6^k + 2^k}$  must also converge by the Direct Comparison Test.