Math 1552: Integral Calculus

Review Problems for Test 1, Sections 4.8, 5.1-5.6, 8.2-8.3

1. **Formula Recap**: complete each of the following formulas.

(a) The general Riemann Sum is found using the formula:

(b) Some helpful summation formulas are:

$$\sum_{i=1}^{n} c =$$

$$\sum_{i=1}^{n} i =$$

$$\sum_{i=1}^{n} i^2 =$$

(c) Properties of the definite integral:

$$\int_a^a f(x)dx =$$

$$\int_b^a f(x)dx =$$

$$\int_a^b cf(x)dx =$$

(d) State the Fundamental Theorem of Calculus:

(e) Using the FTC:

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)dt\right] =$$

(f) If F is an antiderivative of f, that means:

(g) If F is an antiderivative of f, then:

$$\int f(g(x))g'(x)dx =$$

$$\int_a^b f(g(x))g'(x)dx =$$

(h) To find the area between two curves, use the following steps:

(h) Evaluate an integral using *integration by parts* if:

To choose the value of u, use the rule: ______________.

(i) To evaluate integrals with powers or products of trig functions, use the following trig identities to try to obtain a u-substitution:

2. Fill in the integration formulas below:

$$\int x^n dx, \quad (n \neq -1) =$$

$$\int \sin(ax)dx =$$

$$\int \cos(ax)dx =$$

$$\int \sec^2(ax)dx =$$

$$\int \sec(ax)\tan(ax)dx =$$

$$\int \csc(ax)\cot(ax)dx =$$

$$\int \csc^2(ax)dx =$$

$$\int \frac{1}{1+(ax)^2}dx =$$

$$\int \frac{1}{\sqrt{1-(ax)^2}}dx =$$

$$\int \frac{1}{x}dx =$$

$$\int e^{ax}dx =$$

$$\int b^{ax}dx =$$

$$\int \tan x dx =$$

$$\int \sec x dx =$$

$$\int \csc x dx =$$

$$\int \cot x dx =$$

Problems from Recitation Worksheets

1. True or False?

(a) If F and G are both antiderivatives of f, then $F = G$.

(b) The antiderivative of $\sec^2(3x)$ is $\frac{1}{3}\tan(3x)$.

(c) The indefinite integral of a function f is the collection of all antiderivatives of f.

(d) We know how to find the antiderivative of $\cos(x^2)$, and it is $\sin(x^2)$.

(e) To find the upper sum U_f of a function f on $[a, b]$, after partitioning the interval into n pieces, evaluate f at the right-hand endpoint of each subinterval.

(f) When the interval $[a, b]$ is partitioned into n pieces, there are exactly n endpoints.

(g) A partition of the interval $[a, b]$ does not need to be evenly spaced in order to calculate a Riemann Sum.

(h) If f is positive and continuous on $[a, b]$, and A is the actual area bounded by f, $x = a$, $x = b$, and the x-axis, then $L_f < A < U_f$.

(i) We always set x_i^* to be the right-hand endpoint of the i^{th} interval.

(j) $\sum_{i=1}^{n} i^2 = \left(\frac{n(n+1)}{2}\right)^2$.

(k) If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\, dx$ represents the total area bounded by f, $x = a$, $x = b$, and the x-axis.

(l) If f is a continuous function, then the function $F(x) = \int_a^x f(t)dt$ is an anti-derivative of f.

(m) If F is an anti-derivative of f, then $\int_a^b f(x)dx$ represents the slope of the secant line of F(x) on the interval $[a, b]$.

(n) $\frac{d}{dx}\left[\int_a^b f(t)dt\right] = f(x)$.

(o) Given that f is continuous on $[a, b]$ and $F'(x) = f(x)$, then $F(b) - F(a)$ represents the net area bounded by the graph of $y = f(x)$, the lines $x = a$, $x = b$, and the x-axis.

(p) $\int f(x)g(x)\, dx = \left(\int f(x)\, dx\right) \cdot \left(\int g(x)\, dx\right)$

(q) To evaluate $\int \sin^{-1}(x)dx$ by parts, choose $u = \sin^{-1}(x)$ and $dv = dx$.

(r) To evaluate $\int x\ln(x)\, dx$ by parts, choose $u = x$ and $dv = \ln(x)\, dx$.

(s) To evaluate $\int \cot(x)\, dx$, integrate by substitution choosing $u = \sin(x)$.

2. Evaluate the following indefinite integrals.

(a) $\int \left(\sqrt{x} - \frac{1}{x}\right)^2 dx$

(b) $\int \left[4^{-2x} + e^{-5x}\right] dx$

(c) $\int \left(\frac{e^{\sqrt{2}} + x^{\sqrt{2}}}{\sqrt{x}}\right) dx$

(d) $\int \left(\frac{2}{3x} - \frac{1}{\sqrt{4-x^2}}\right) dx$

3. A particle travels with a velocity given by $v(t) = -\frac{1}{3}t^2 + 4t + 2$, where position is measured in meters and time in seconds.

(a) Find the acceleration of the particle when $t = 1$ second.

(b) If the initial position is 4 m, find the position of the particle at $t = 1$ second.

4. (*Applying the Riemann Sum*) You are driving when all of a sudden, you see traffic stopped in front of you. You slam the brakes to come to a stop. While your brakes are applied, the velocity of the car is measured, and you obtain the following measurements:

Time since applying breaks (sec)	0	1	2	3	4	5
Velocity of car (in ft/sec)	88	60	40	25	10	0

(a) Plot the points on a curve of velocity vs. time.

(b) Using the points given, determine upper and lower bounds for the total distance traveled before the car came to a stop.

5. Estimate the area under the graph of $f(x) = 10 - x^2$ between the lines $x = -3$ and $x = 2$ using $n = 5$ equally spaced subintervals, by finding:

(a) the upper sum, U_f.

(b) the lower sum, L_f.

6. (*Applying the Definite Integral*) A marketing company is trying a new campaign. The campaign lasts for three weeks, and during this time, the company finds that it gains customers as a function of time according to the formula:

$$C(t) = 5t - t^2,$$

where t is time in weeks and the number of customers is given in thousands.

Using the general form of the definite integral,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*),$$

calculate the **average** number of customers gained during the three-week campaign.

7. Explain why the following property is true:

$$|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx.$$

Can you find an example where the inequality is strict?

8. Using the general form of the definite integral, $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$, evaluate:

$$\int_2^4 (x-1)^2 dx.$$

9. Evaluate $\int_0^2 |x-1|dx$ using integral properties from class. (HINT: draw a picture, and use geometry!)

10. Suppose that $f(x)$ is an even function such that $\int_0^2 f(x)dx = 5$ and $\int_0^3 f(x)dx = 8$. Find the value of $\int_{-2}^{3} f(x)dx$.

11. Evaluate the integrals:
(a) $\int_1^2 \frac{3x-5}{x^3} dx$.
(b) $\int_2^5 (2-x)(x-5)dx$.
(c) $\int_\pi^{\frac{7\pi}{2}} \frac{1+\cos(2t)}{2} dt$.

12. Find $F'(2)$ for the function

$$F(x) = \int_{\frac{8}{x}}^{x^2} \left(\frac{t}{1-\sqrt{t}}\right) dt.$$

13. (a) Given the function f below, evaluate $\int_1^9 f(x)dx$.

$$f(x) = \begin{cases} x^2+4, & x<4 \\ \sqrt{x}-x, & x \geq 4 \end{cases}$$

(b) Would you get the same answer to part (a) if you evaluated $F(9) - F(1)$? What does this tell you about the FTC and continuity?

14. (a) Evaluate the expressions:

$$\int_0^1 x(1+x)\,dx, \quad \left(\int_0^1 x\,dx \cdot \int_0^1 (1+x)\,dx\right)$$

(b) Looking at your answer in part (a), what, if anything, can you say in general about $\int (f(x) \cdot g(x))dx$?

15. For each integral below, determine if we can evaluate the integral using the method of u-substitution. If the answer is "yes", evaluate the integral.

(a) $\int \frac{1}{x^2} \sec\left(\frac{1}{x}\right) \tan\left(\frac{1}{x}\right) dx$

(b) $\int x \csc^2(x) dx$

(c) $\int \frac{\sin 3x - \cos 3x}{\sin 3x + \cos 3x} dx$

(d) $\int \tan(x^2) dx$

16. Evaluate the following integrals using the method of substitution.

(a) $\int \frac{1}{\ln(x^x)} dx$

(b) $\int \frac{e^{2x}}{\sqrt{4-3e^{2x}}} dx$

(c) $\int \frac{dx}{\sqrt{4-(x+3)^2}}$

17. Suppose that $y = f(x)$ and $y = g(x)$ are both continuous functions on the interval $[a, b]$. Determine if each statement below is always true or sometimes false.

(a) Suppose that $f(c) > g(c)$ for some number $c \in (a, b)$. Then the area bounded by f, g, $x = a$, and $x = b$ can be found by evaluating the integral $\int_a^b (f(x) - g(x))\,dx$.

(b) If $\int_a^b (f(x) - g(x))\,dx$ evaluates to -5, then the area bounded by f, g, $x = a$, and $x = b$ is 5.

(c) If $f(x) > g(x)$ for every $x \in [a, b]$, then $\int_a^b |f(x) - g(x)|\,dx = \int_a^b (f(x) - g(x))\,dx$.

18. Find the area bounded by the region between the curves $f(x) = x^3 + 2x^2$ and $g(x) = x^2 + 2x$.

19. Find the area bounded by the region enclosed by the three curves $y = x^3$, $y = -x$, and $y = -1$.

20. Find the area bounded by the curves $y = \cos x$ and $y = \sin(2x)$ on the interval $\left[0, \frac{\pi}{2}\right]$.

21. Find the area of the triangle with vertices at the points (0,1), (3,4), and (4,2). USE CALCULUS.

22. For each function below: (i) determine which method to use to evaluate the function (formula, u-substitution, or integration by parts, and (ii) evaluate the integral.

(a) $\int_1^e \frac{\sqrt{\ln x}}{x}\, dx$
(b) $\int (\ln x)^2\, dx$
(c) $\int x^2 e^{x^3}\, dx$
(d) $\int x^3 e^{x^2}\, dx$
(e) $\int 4^{-x}\, dx$
(f) $\int x^2 \cdot 4^x\, dx$

23. Determine if each integral below can be evaluated using a method we have learned so far (formula, u-substitution, integration by parts, or trig identities). If so, evaluate the integral. If not, explain why it cannot be evaluated.

(a) $\int x^5 \ln(x)\, dx$
(b) $\int \sin^5(2x)\cos^3(2x)\, dx$
(c) $\int \cos^2(3x)\, dx$
(d) $\int \tan(x)\ln[\cos(x)]\, dx$
(e) $\int \sin\left(x^2\right)\, dx$
(f) $\int \tan^4(x) dx$
(g) $\int e^{2x}\sin(3x) dx$

Additional Test Review Problems

24. True or false?

(a) When evaluating a **definite** integral using u-subsitution, different choices of u may lead to different final answers.

(b) Integration by Parts is a Product Rule in integral form.

(c) The goal of integration by parts is to go from an integral $\int f'(x)g'(x)dx$ that we can't evaluate to an integral $\int f(x)g(x)dx$ that we can evaluate.

(d) Definite integrals can not be evaluated by Integration by Parts.

(e) If f is a continuous, increasing function, then the right-hand Riemann sum method always overestimates the definite integral.

(f) Let f be a continuous function and $av(f)$ be the average of f. Then $av(f) \cdot (b-a) = \int_a^b f(x)dx$.

(g) When finding the area between the curves $y = x^3 - x$ and $y = x^2 + x$ it suffices to find the value of the definite integral $\int_{-1}^{2} \left[(x^3 - x) - (x^2 + x)\right] \, dx$, and then take the absolute value of this value to get the right answer.

(h) To find the area between the curves $y = x^3 - x$ and $y = x^2 + x$, first set the equations equal and solve to find the intersection points $x = -1$ and $x = 2$, plug in a test-point into the equations or graph the curves to determine **top** and **bot**, and then evaluate $\int_{-1}^{2} (\mathbf{top}) - (\mathbf{bot}) \, dx$.

(i) If $\int_0^1 f(x) \, dx = 9$ and $f(x) \geq 0$, then $\int_0^1 \sqrt{f(x)} \, dx = 3$.

25. Evaluate the following integrals.

(a) $\int_0^{\frac{\pi}{4}} \sec^2(t) e^{1+\tan(t)} \, dt$

(b) $\int \sin^3(x) \cos^3(x) \, dx$

(c) $\int \frac{1}{\sqrt{4-9w^2}} \, dw$

(d) $\int x \sin(x) \cos(x) \, dx$

(e) $\int \sec^4(x) \, dx$

(f) $\int \ln(x+1) \, dx$

26. Suppose: $f(1) = 2, f(4) = 7, f'(1) = 5, f'(4) = 3$ and $f''(x)$ is continuous. Find the

value of:

$$\int_1^4 f''(x)\,dx.$$

27. Consider the following limit

$$\lim_{n\to\infty}\sum_{i=1}^{n}\cos\left(\pi\cdot\frac{i}{n}\right)\cdot\frac{\pi}{2n}.$$

(a) Express the limit as a definite integral.

(b) Compute the definite integral from part (a).

28. Let $f(x) = 3x + 4$.

(a) Estimate the area of the region between the graph of f, the lines $x = -1$ and $x = 2$, and the x-axis using a upper sum with three rectangles of equal width.

(b) Find the actual area in part (a) by taking the limit of a general Riemann Sum using n equally spaced subintervals, and taking x_i^* as the right-hand endpoint of each interval.

29. Find the area bounded by the curves $y = \cos^2(x)$ and $y = -\sin^2(x)$, and the lines $x = 0$ and $x = \pi$. (Hint: draw a picture in GeoGebra - an online graphing tool.)

30. Find the area bounded by the curves $y = -x^2 + 6x$ and $y = x^2 - 2x - 24$. (Hint: sketch the curves or make a sign chart.)

31. Find $F'(4)$ if

$$F(x) = \int_{\frac{x^2}{4}}^{x^2} \ln(\sqrt{t})\,dt.$$

32. What value of $b > -1$ maximizes the integral:

$$\int_{-1}^{b} x^2(7-x)dx?$$

33. Find a number c so that $f(c)$ is equal to the average value of the function $f(x) = 1+x$ on the interval $[-1, 3]$. Graphically, what does that mean?

Answers

1. (b), (c), (g), (k), (l), (o), (q), (s) are true

2. (a) $\frac{1}{2}x^2 - 4\sqrt{x} - \frac{1}{x} + C$

(b) $-\frac{1}{2\ln 4}4^{-2x} - \frac{1}{5}e^{-5x} + C$

(c) $2e^{\sqrt{2}}\sqrt{x} + \frac{1}{\sqrt{2}+1/2}x^{\sqrt{2}+1/2} + C$

(d) $\frac{2}{3}\ln|x| - \sin^{-1}\left(\frac{x}{2}\right) + C$

3. (a) $\frac{10}{3}$ m/s^2, (b) $7\frac{8}{9}$ m

4. (b) Upper: 223 ft, Lower: 135 ft

5. (a) 44 (b) 31

6. 4,500 customers

7. Consider the difference between NET and TOTAL area.

8. $\frac{26}{3}$

9. 1

10. 13

11. (a) $-\frac{3}{8}$; (b) $\frac{9}{2}$; (c) $\frac{5\pi}{4}$

12. -24

13. (a) $\frac{79}{6}$; (b) you cannot use the FTC as stated when f is discontinuous somewhere on the interval $[a, b]$

14. (a) $\frac{5}{6}$ and $\frac{3}{4}$; no general rule

15. (a) $-\sec\left(\frac{1}{x}\right) + C$, (c) $-\frac{1}{3}\ln|\sin 3x + \cos 3x| + C$

16. (a) $\ln|\ln x| + C$, (b) $-\frac{1}{3}\sqrt{4 - 3e^{2x}} + C$, (c) $\sin^{-1}\left(\frac{x+3}{2}\right) + C$

17. (c) is true

18. $\frac{37}{12}$

19. $\frac{5}{4}$

20. $\frac{1}{2}$

21. 5. 4.5

22. (a) $\frac{2}{3}$

(b) $x(\ln x)^2 - 2x\ln x + 2x + C$

(c) $\frac{1}{3}e^{x^3} + C$

(d) $\frac{x^2e^{x^2}}{2} - \frac{e^{x^2}}{2} + C$

(e) $-\frac{1}{\ln 4}4^{-x} + C$

(f) $\frac{1}{\ln 4}x^2 \cdot 4^x - \frac{2}{(\ln 4)^2}x \cdot 4^x + \frac{2}{(\ln 4)^3}4^x + C$

23. (a) $\frac{x^6 \ln x}{6} - \frac{x^6}{36} + C$

(b) $\frac{1}{12}\sin^6(2x) - \frac{1}{16}\sin^8(2x) + C$

(c) $\frac{1}{2}x + \frac{1}{12}\sin(6x) + C$

(d) $-\frac{1}{2}(\ln[\cos(x)])^2 + C$

(e) Cannot be evaluated

(f) $\frac{1}{3}\tan^3(x) - \tan(x) + x + C$

(g) $\frac{2}{13}e^{2x}\sin(3x) - \frac{3}{13}e^{2x}\cos(3x) + C$

24. (e), (f) are true

25. (a) $e^2 - e$

(b) $\frac{1}{6}\cos^6(x) - \frac{1}{4}\cos^4(x) + C$

(c) $\frac{1}{3}\arcsin\left(\frac{3w}{2}\right) + C$

(d) $\frac{x}{2}\sin^2 x - \frac{1}{4}x + \frac{1}{8}\sin 2x + C$

(e) $\tan(x) + \frac{\tan^3(x)}{3} + C$

(f) $(x+1)\ln(x+1) - (x+1) + C$

26. 2

27. (a) $\int_0^{\frac{\pi}{2}} \cos(2x)dx$, (b) 0

28. (a) 21, (b) 16.5

29. π

30. $\frac{512}{3}$ or approximately 170.67

31. $14\ln 2$

32. $b = 7$

33. $c = 1$