Math 1552, Integral Calculus

Review for Test 2

Sections 7.2, 8.2-8.5, 4.5

1. Content Recap

(a) To apply L'Hopital's rule, the limit must have the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

(b) A *separable differential equation* has the general form:

$$\frac{dy}{dx} = p(x)q(y)$$

To solve this equation, use the following steps:

Separate the variables and integrate:

$$\int \frac{1}{q(y)}\,dy = \int p(x)\,dx$$

(c) Evaluate an integral using *integration by parts* if:

it contains a product of different types of functions, or a log/inverse that can't be solved with u-sub

To choose the value of u, use the rule: ILATE.

(d) To evaluate integrals with powers or products of trig functions, use the following trig identities to try to obtain a u-substitution:

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$
$$\sin^2 x = \frac{1}{2}\left[1 - \cos(2x)\right]$$
$$\cos^2 x = \frac{1}{2}\left[1 + \cos(2x)\right]$$
$$\sin(2x) = 2\sin x\cos x$$

(e) If we would evaluate an integral using *trig substitution*, the integral should contain an expression of one of these forms: $a^2 - x^2$, $a^2 + x^2$, or $x^2 - a^2$.

Write out the trig substitution you would use for each form listed above.

$a^2 - x^2$, set $x = a\sin\theta$

$a^2 + x^2$, set $x = a\tan\theta$

$x^2 - a^2$, set $x = a\sec\theta$

(f) To use the method of *partial fractions*, we must first factor the denominator completely into linear or irreducible quadratic terms.

In the partial fraction decomposition, if the term in the denominator is raised to the kth power, then we have K partial fractions.

For each linear term, the numerator of the partial fraction will be a constant

For each irreducible quadratic term, the numerator will be linear.

2. Evaluate each integral below using any of the methods we have learned.

(a) $\int \frac{\sin^3 x}{\cos x}dx$

(b) $\int \frac{x}{\sqrt{x^2+2x-3}}dx$

(c) $\int x\tan^{-1}(x^2)dx$

(d) $\int \frac{\cos x}{4+\sin^2 x}dx$

(e) $\int \cos[\ln(x)]dx$

(f) $\int \frac{1}{x(x^2+x+1)}dx$

(a) $\int \frac{\sin^3 x}{\cos x}dx = \int \frac{\sin^2 x}{\cos x}\cdot \sin x\,dx = \int \frac{1-\cos^2 x}{\cos x}\cdot \sin x\,dx$

$= \int \left(\frac{1}{\cos x} - \cos x\right)\cdot \sin x\,dx \underset{du=-\sin x\,dx}{\overset{u=\cos x}{=}} -\int \left(\frac{1}{u}-u\right)du$

$= \boxed{-\ln|\cos x| + \frac{1}{2}\cos^2 x + C}$

(b) $\int \frac{x}{\sqrt{x^2+2x-3}}dx = \int \frac{x}{\sqrt{x^2+2x+1-4}}dx$

$= \int \frac{x}{\sqrt{(x+1)^2-4}}dx$

Trig sub: Let $x+1 = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

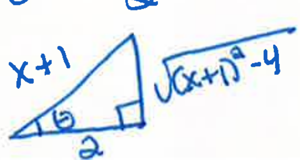
and $x = 2\sec\theta - 1$

$= \int \frac{2\sec\theta - 1}{\sqrt{4\sec^2\theta-4}}\cdot 2\sec\theta\tan\theta\,d\theta = \int \frac{2\sec\theta-1}{\cancel{2}\cancel{\tan\theta}}\cdot \cancel{2}\sec\theta\,\cancel{\tan\theta}\,d\theta$

$= \int (2\sec^2\theta - \sec\theta)\,d\theta = 2\tan\theta - \ln|\sec\theta+\tan\theta| + C$

$= 2\cdot\frac{\sqrt{(x+1)^2-4}}{2} - \ln\left|\frac{x+1}{2} + \frac{\sqrt{(x+1)^2-4}}{2}\right| + C$

$\sec\theta = \frac{x+1}{2} = \frac{h}{a}$



$= \boxed{\sqrt{x^2+2x-3} - \ln\left|\frac{x+1}{2} + \frac{\sqrt{x^2+2x-3}}{2}\right| + C}$

3. Find the general solution to the equation:

$$(y \ln x)y' = \frac{y^2+1}{x}.$$

$$\int \frac{y}{y^2+1}\,dy = \int \frac{1}{x\ln x}\,dx \Rightarrow \frac{1}{2}\ln(y^2+1) = \ln|\ln x| + C$$

$$\ln(y^2+1) = 2\ln|\ln x| + 2C$$

$$y^2+1 = e^{2\ln|\ln x| + 2C} \qquad \text{let } K = e^{2C}$$

$$\boxed{y^2+1 = Ke^{2\ln|\ln x|}}$$

4. A radioactive substance loses 20% of its mass each year. Find the half-life (i.e., the time it requires to have half of the substance remaining) of the substance.

$$A(1) = 0.8A_0 = A_0e^{k} \Rightarrow e^{k} = 0.8, \text{ so}$$

$$A(t) = A_0(0.8)^t$$

Solve $A(t) = \frac{1}{2}A_0 \Rightarrow \cancel{A_0}(0.8)^t = 0.5\cancel{A_0}$

$$t\ln(0.8) = \ln(0.5)$$

$$t = \frac{\ln(0.5)}{\ln(0.8)} \approx 3.106 \text{ years}$$

5. Evaluate the following limits:

$$\lim_{x\to 0} \frac{\ln(\sec x)}{x^2}, \qquad \lim_{x\to\infty}\left(\cos\frac{1}{x}\right)^x.$$

(a)

$$\lim_{x\to 0} \frac{\ln(\sec x)}{x^2} \left[\frac{0}{0}\right] \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1}{\sec x}\cdot \sec x\tan x}{2x}$$

$$= \lim_{x\to 0} \frac{\tan(x)}{2x} \left[\frac{0}{0}\right] \overset{L'H}{=} \lim_{x\to 0} \frac{\sec^2 x}{2} = \boxed{\frac{1}{2}}$$

(b) See back of last page

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(c) $\int x \tan^{-1}(x^2)\,dx$

By parts: Let $u = \tan^{-1}(x^2)$ $\quad \int dv = \int x\,dx$

$du = \frac{2x}{1+x^4}dx \quad v = \frac{1}{2}x^2$

$$\int x\tan^{-1}(x^2)\,dx = \frac{1}{2}x^2\tan^{-1}(x^2) - \frac{1}{2}\int \frac{2x^3}{1+x^4}\,dx$$

Let $w = 1+x^4$

$dw = 4x^3\,dx$

$\frac{1}{2}dw = 2x^3\,dx$

$$= \frac{1}{2}x^2\tan^{-1}(x^2) - \frac{1}{4}\int \frac{dw}{w}$$

$$= \boxed{\frac{1}{2}x^2\tan^{-1}(x^2) - \frac{1}{4}\ln(1+x^4) + C}$$

(d) $\int \frac{\cos x}{4+\sin^2 x}\,dx = \frac{1}{4}\int \frac{\cos x}{1+\left(\frac{\sin x}{2}\right)^2}\,dx$

Let $u = \frac{\sin x}{2}$, $du = \frac{1}{2}\cos x\,dx$

$$= \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C = \boxed{\frac{1}{2}\tan^{-1}\left(\frac{\sin x}{2}\right) + C}$$

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(e) $\int \cos(\ln(x))\,dx = I$

By parts: Let $u = \cos(\ln x)$ $\quad \int dv = \int dx$

$du = -\frac{\sin(\ln x)}{x}\,dx \quad v = x$

$I = x\cos(\ln x) + \int \sin(\ln x)\,dx$

By parts again: $u' = \sin(\ln x)$ $\quad \int dv' = \int dx$

$du' = \frac{\cos(\ln x)}{x}\,dx \quad v' = x$

$I = x\cos(\ln x) + x\sin(\ln x) - \underbrace{\int \cos(\ln x)\,dx}_{I}$

$2I = x\cos(\ln x) + x\sin(\ln x)$

$\boxed{I = \frac{1}{2}\left[x\cos(\ln x) + x\sin(\ln x)\right] + C}$

(f) $\int \frac{1}{x(x^2+x+1)}\,dx$

Partial Fractions

$$\frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

$1 = A(x^2+x+1) + (Bx+C)x$

$1 = (A+B)x^2 + (A+C)x + A$

$A+B = 0 \Rightarrow B = -1$

$A+C = 0 \Rightarrow C = -1$

$A = 1$

#2 (f)

$$\int \frac{1}{x(x^2+x+1)}\,dx = \int \left[\frac{1}{x} - \frac{x+1}{x^2+x+1}\right] dx$$

$$= \int \frac{1}{x}\,dx - \int \frac{x+1}{x^2+x+1}\,dx$$

$$= \ln|x| - \frac{1}{2}\int \frac{2\left(x+1+\frac{1}{2}-\frac{1}{2}\right)}{x^2+x+1}\,dx$$

$$= \ln|x| - \frac{1}{2}\int \frac{2\left(x+\frac{1}{2}+\frac{1}{2}\right)}{(x^2+x+1)}\,dx$$

$$= \ln|x| - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{dx}{x^2+x+1}$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+x+1| - \frac{1}{2}\int \frac{dx}{\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}}$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+x+1| - \frac{1}{2}\int \frac{dx}{(x+1/2)^2+3/4}$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+x+1| - \frac{2}{3}\int \frac{dx}{\left(\frac{x+1/2}{\sqrt{3}/2}\right)^2+1}$$

$u = \frac{x+1/2}{\sqrt{3}/2}$

$du = 2/\sqrt{3}\,dx$

$\frac{\sqrt{3}}{2}du = dx$

$$= \ln|x| - \frac{1}{2}\ln|x^2+x+1| - \frac{2}{3}\cdot\frac{\sqrt{3}}{2}\tan^{-1}(u) + C$$

$$= \boxed{\ln|x| - \frac{1}{2}\ln|x^2+x+1| - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C}$$

#5 part (b):

$$\lim_{x\to\infty}\left(\cos\frac{1}{x}\right)^x \quad [1^\infty]$$

Let $y=\left(\cos\frac{1}{x}\right)^x$, then

$$\ln y = x\ln\left(\cos\tfrac{1}{x}\right)$$

$$= \frac{\ln\left(\cos\frac{1}{x}\right)}{1/x}$$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln\left(\cos\frac{1}{x}\right)}{\frac{1}{x}} \quad \left[\frac{0}{0}\right]$$

$$\stackrel{L'H}{=}\lim_{x\to\infty}\frac{\frac{1}{\cos\frac{1}{x}}\cdot\left(-\sin\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$=\lim_{x\to\infty}-\tan\frac{1}{x} = 0,$$

so $\lim_{x\to\infty} y = e^0 = \boxed{1.}$