Section 1.3

Parametric Form

https://textbooks.math.gatech.edu/ila/demos/rrinter.html?mat=2,1,12,1:1,2,9,-1

§1.3

Example 1

How do we solve a system of linear equations if the row reduced matrix has a column without a pivot? Let's do an example.

$$\begin{aligned} 2x + y + 12z &= 1 \\ x + 2y + 9z &= -1 \end{aligned} \quad \text{gives rise to the matrix} \quad \left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right)$$

How do you write down the solutions to a system of linear eqns?

① $[A|b]$

② $[A|b] \sim \cdots$ REF/RREF

③ write eqns of soln

① NOT equality

RREF

$$\left[\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\begin{cases} x + 5z = 1 \\ y + 2z = -1 \end{cases} \Rightarrow \boxed{\begin{cases} x = 1 - 5z \\ y = -1 - 2z \\ z = z \quad \text{(free)} \end{cases}} \quad (z \text{ in } \mathbb{R})$$

solutions to the original system written in parametric equation form

ex. some particular solns are

$(z=0)$	$(z=1)$	$(z=-1)$
$x=1$	$x=-4$	$x=6$
$y=-1$	$y=-3$	$y=1$
$z=0$	$z=1$	$z=-1$

$\leftarrow \cdots$ lots more

Describe this soln set geometrically

ans: a line in $\mathbb{R}^3$

(one free var)

(3 variables in eqns)

Free Variables

Definition

Consider a *consistent* linear system of equations in the variables $x_1, \ldots, x_n$. Let A be a row echelon form of the matrix for this system.

We say that x_i is a **free variable** if its corresponding column in A is *not* a pivot column.

> **Important**
>
> 1. You can choose *any value* for the free variables in a (consistent) linear system.
> 2. Free variables come from *columns without pivots* in a matrix in row echelon form.

x, y, z, w vars.

Example 2

Suppose the reduced row echelon form of the matrix for a linear system in x_1, x_2, x_3, x_4 is

$$\boxed{\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right)}$$

RREF

What happened to x_2? What is it allowed to be?

> The boxed equation is called the **parametric form** of the general solution to the system of equations. It is obtained by moving all free variables to the right-hand side of the $=$.

① $\begin{cases} x + 3w = 2 \\ z + 4w = -1 \end{cases}$ ② $\Rightarrow$

param. eqn. form

$$\boxed{\begin{aligned} x &= 2 - 3w \\ y &= y \quad \text{(free)} \\ z &= -1 - 4w \\ w &= w \quad \text{(free)} \end{aligned}}$$

$2 + 0(1) + 0(-1) + 3(0) = 2$ ✓

$\begin{cases} 1x + 0y + 0z + 3w = 2 \\ 0x + 0y + 1z + 4w = -1 \end{cases}$

① $[A|b]$ ✓

② $[A|b] \sim \cdots$ REF/RREF ✓

③ & ④ write easier system and solve

list some solutions

$(y=0,\ w=0)$	$(y=1,\ w=0)$	$(y=1,\ w=2)$
$x=2$	$x=2$	$x=-4$
$y=0$	$y=1$	$y=1$
$z=-1$	$z=-1$	$z=-9$
$w=0$	$w=0$	$w=2$

Example 3

Solve the system of linear equations in x_1, x_2, x_3, x_4:

$$\begin{aligned} x_1 \quad + 5x_3 \quad &= 0 \\ x_4 &= 0 \end{aligned}$$

$$\begin{array}{cccccc} & x_2 & x_3 & \text{free} & & \\ & \downarrow & \downarrow & & & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

RREF ✓

$$\boxed{\begin{cases} x_1 = -5x_3 \\ x_2 = x_2 \quad \text{(free)} \\ x_3 = x_3 \quad \text{(free)} \\ x_4 = 0 \end{cases}}$$

META.

① $[A|b]$ ✓

② $[A|b] \sim \cdots$ REF/RREF ✓

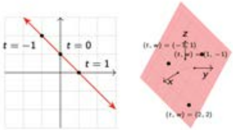
③ & ④ write easier system and solve

Free variables

Geometry, looking ahead!

If we have a consistent system of linear equations, with n variables and k free variables, then the set of solutions is a k-dimensional object in $\mathbb{R}^n$. We will make this precise later, but it is worth thinking about now.

Why does this make sense?



You try it!

Poll

A linear system has 4 variables and 3 equations.

What are the possible solution sets?

(a) point
(b) two points
(c) line
(d) plane
(e) 3-dimensional object
(f) 4-dimensional object

coefficient matrix
3 rows & 4 cols

$$[A|b] = \left[\begin{array}{cccc|c} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}\right]$$

augmented matrix
3 rows & 5 cols.

(f) $\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $0=0$, $0=0$, $0=0$ $x_1 = $ free, $x_2 = $ free, $x_3 = $ free, $x_4 = $ free 4-dim'l solution set

(a) impossible. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$ can't have 4 pivots
at most 3 pivots
(b/c only 3 eqn)
at least one free var.

(e) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = 2$, $x_2 = $ free, $x_3 = $ free, $x_4 = $ free 3-dim'l solution set.

(b) only possible # of sol'ns are
* one unique soln,
* infinitely many, or
* no soln.

(d) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = 2$, $x_2 = 3$, $x_3 = $ free, $x_4 = $ free 2-dim'l solution set

z is free

(c) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$ works

(c) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \end{array}\right]$ $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = $ free

z, w both free

(d) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $x = 0$, $y = 0$, $z = z$ (free), $w = w$ (free)

(a) not possible b/c # eqns > # vars
so couldn't have any pivot var
(at most 1 pivot per row).

y, z, w both free

(d) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $x = 0$, $y = y$ (free), $z = z$ (free), $w = w$ (free)

(b) NEVER POSSIBLE.
only possible # of sol'ns
none/exactly one/infinitely many

Yet Another Example

The linear system

$x + y + z = 1$ has matrix form $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right)$.

Q: What is the parametric form of the general solution to this linear system of one equation in three variables?

Trichotomy

There are *three possibilities* for the reduced row echelon form of the augmented matrix of a linear system.

1. The last column is a pivot column.
   In this case, the system is *inconsistent*. There are *zero* solutions, i.e. the solution set is *empty*. Picture:

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

2. Every column except the last column is a pivot column.
   In this case, the system has a *unique solution*. Picture:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right)$$

$x=a$
$y=b$
$z=c$

unique soln!

3. The last column is not a pivot column, and some other column isn't either.
   In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). Picture:

$$\left(\begin{array}{cccc|c} 1 & * & 0 & * & * \\ 0 & 0 & 1 & * & * \end{array}\right)$$

list them

use

So far.

* Systems of linear eqns.
* Matrices & row reducing
* representing solns using $\boxed{\text{parametric eqn. form.}}$

Summary

- **Row reduction** is an algorithm for solving a system of linear equations represented by an augmented matrix.
- The goal of row reduction is to put a matrix into **(reduced) row echelon form**, which is the "solved" version of the matrix.
- An augmented matrix corresponds to an inconsistent system if and only if there is a pivot in the augmented column.
- Columns without pivots in the RREF of a matrix correspond to **free variables**. You can assign any value you want to the free variables.
- We can tell whether a linear system has zero, one, or infinitely many solutions using the RREF of the augmented matrix.

Find the solutions to the system of linear equations with **augmented** matrix given below.

(a) $$\begin{bmatrix} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{bmatrix} \sim$$

(b) $$\begin{bmatrix} 10 & -3 & -2 & 7 \end{bmatrix}$$

Chapter 2

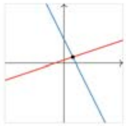
Systems of Linear Equations: Geometry

Motivation

We want to think about the *algebra* in linear algebra (systems of equations and their solution sets) in terms of *geometry* (points, lines, planes, etc).

algebraic

$$\begin{aligned} x - 3y &= -3 \\ 2x + y &= 8 \end{aligned} \rightsquigarrow$$



vs. geometry.

Section 2.1

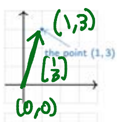
Vectors

Points and Vectors

We have been drawing elements of $\mathbf{R}^n$ as points in the line, plane, space, etc. We can also draw them as arrows.

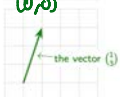
Definition

A **point** is an element of $\mathbf{R}^n$, drawn as a point (a dot).



A **vector** is an element of $\mathbf{R}^n$, drawn as an arrow. When we think of an element of $\mathbf{R}^n$ as a vector, we'll usually write it vectically, like a matrix with one column:

$$v = \begin{pmatrix}1\\3\end{pmatrix}.$$



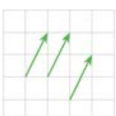
[interactive]

The difference is purely psychological: *points and vectors are just lists of numbers.*

Points and Vectors

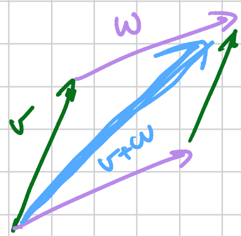
So why make the distinction?

A vector need not start at the origin: *it can be located anywhere!* In other words, an arrow is determined by its length and its direction, not by its location.



These arrows all represent the vector $\begin{pmatrix}1\\2\end{pmatrix}$.

However, unless otherwise specified, we'll assume a vector starts at the origin.



Vector Algebra

Definition

- We can add two vectors together:

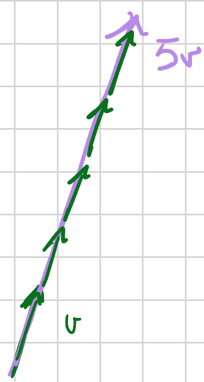
$$\begin{pmatrix}a\\b\\c\end{pmatrix} + \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a+x\\b+y\\c+z\end{pmatrix}.$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}-1\\2\\5\end{bmatrix} = \begin{bmatrix}0\\4\\8\end{bmatrix}$$

- We can multiply, or **scale**, a vector by a real number c:

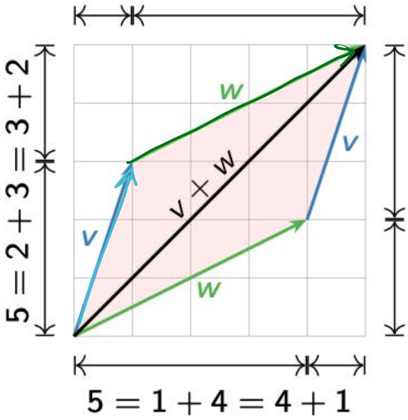
$$c\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}c\cdot x\\c\cdot y\\c\cdot z\end{pmatrix}.$$

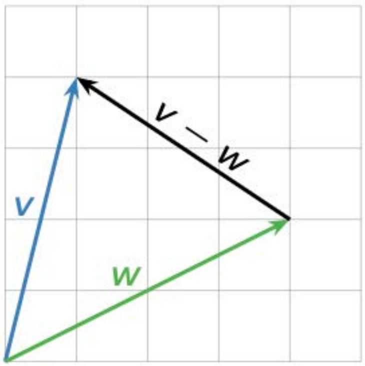
$$5\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}5\\10\\15\end{bmatrix}$$



We call c a **scalar** to distinguish it from a vector. If v is a vector and c is a scalar, cv is called a **scalar multiple** of v.

Vector Addition and Subtraction: Geometry

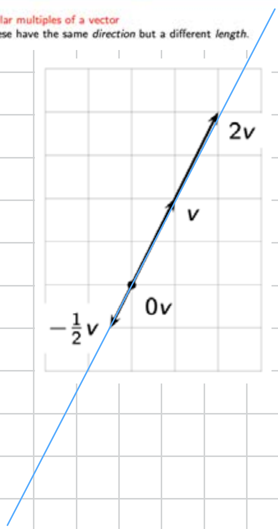




$\cancel{w} + (v - \cancel{w}) = v$

Scalar Multiplication: Geometry

Scalar multiples of a vector
These have the same *direction* but a different *length*.



Linear Combinations

We can add and scalar multiply in the same equation:

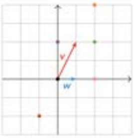
$$w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

where $c_1, c_2, \ldots, c_p$ are scalars, $v_1, v_2, \ldots, v_p$ are vectors in $\mathbf{R}^n$, and w is a vector in $\mathbf{R}^n$.

Definition

We call w a **linear combination** of the vectors $v_1, v_2, \ldots, v_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the **weights** or **coefficients**.

Example



Let $v = \begin{pmatrix}1\\2\end{pmatrix}$ and $w = \begin{pmatrix}1\\0\end{pmatrix}$.

What are some linear combinations of v and w?

- $v + w$
- $v - w$
- $2v + 0w$
- $2w$
- $-v$

[interactive: 2 vectors] [interactive: 3 vectors]

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https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,1,-1&v3=-1,1,4&range=5&captions=combo&nomove=true

You Try It!

Example. If v=[1;2] and w=[1;0] then find

- v+w $= \begin{bmatrix}2\\2\end{bmatrix}$
- v-w $= \begin{bmatrix}0\\2\end{bmatrix}$
- 2v+0w $= 2v = 2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$
- 2w $= \begin{bmatrix}2\\0\end{bmatrix}$
- -v $= \begin{bmatrix}-1\\-2\end{bmatrix}$

Poll

Poll

Is there any vector in $\mathbf{R}^2$ that is *not* a linear combination of v and w?

$$\begin{bmatrix}a\\b\end{bmatrix} = x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}1\\0\end{bmatrix}$$

This will always work
(can always find x, y scalars that work)
no matter how a,b is chosen.

Poll

Poll

Is there any vector in $\mathbf{R}^2$ that is *not* a linear combination of v and w?

More Examples

What are some linear combinations of $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

What are some linear combinations of $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

$$3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 5 \end{bmatrix}, \begin{bmatrix} 18 \\ 9 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

$$v + 2w = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 0v + 0w = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 2v + 5w = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

every possible like $\begin{bmatrix} a \\ a \end{bmatrix}$

$$x\begin{bmatrix} 2 \\ 2 \end{bmatrix} + y\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{pmatrix} 2x - y \\ 2x - y \end{pmatrix} = (2x-y)\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Section 2.2

Vector Equations and Spans

Systems of Linear Equations

Solve the following system of linear equations:

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3. \end{aligned}$$

We can write all three equations at once as vectors:

$$\begin{pmatrix} x - y \\ 2x - 2y \\ 6x - y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

We can write this as a linear combination:

$$x\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

So we are asking:

Question: Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

$$\begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix} = \begin{bmatrix} x-y \\ 2x-2y \\ 6x-y \end{bmatrix} = \begin{bmatrix} x \\ 2x \\ 6x \end{bmatrix} + \begin{bmatrix} -y \\ -2y \\ -y \end{bmatrix}$$

$$= x\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y\begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$$

Summary

The **vector equation**

$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = b,$$

where $v_1, v_2, \ldots, v_p, b$ are vectors in $\mathbf{R}^n$ and $x_1, x_2, \ldots, x_p$ are scalars, has the same solution set as the linear system with augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array}\right),$$

where the v_i's and b are the columns of the matrix.

This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

Definition

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$. In symbols:

"the set of" "such that"

$$\boxed{\mathrm{Span}\{v_1, v_2, \ldots, v_p\} = \{x_1v_1 + x_2v_2 + \cdots + x_pv_p \mid x_1, x_2, \ldots, x_p \text{ in } \mathbf{R}\}.}$$

Synonyms: $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is the subset **spanned by** or **generated by** $v_1, v_2, \ldots, v_p$.

Span{v, w} is the set of all linear combinations of v & w

Now we have several equivalent ways of making the same statement:

1. A vector b is in the span of $v_1, v_2, \ldots, v_p$.
2. The vector equation
$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = b$$
has a solution.
3. The linear system with augmented matrix
$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array}\right)$$
is consistent.

Sanity check

$$1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} ✓✓✓?$$

Example: Is the vector b in the span of v and w, where b=[1;2;1], v=[1;0;-1] and w=[0;1;1]?

Q: Is $b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ in the span of $v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ & $w = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$?

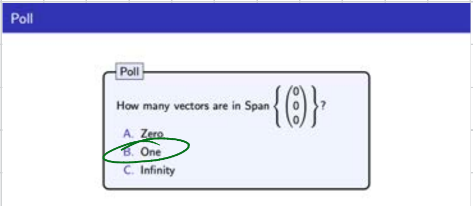
a.k.a. $x\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$\left\{\begin{array}{l} x=1 \\ y=2 \\ 0=0 \end{array}\right.$$

$$\Rightarrow \begin{bmatrix} x+0y \\ y \\ -x+y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \left\{\begin{array}{l} x=1 \\ y=2 \\ -x+y=1 \end{array}\right. \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

system of lin. eqns

You Try It!



$1\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ ✓

$0\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ ✓

$-5\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ ✓

Example: If v1=[2;1], what is Span{v1,3v1}?

Q_1: $\text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}6\\3\end{bmatrix}\right\}$ what is it?

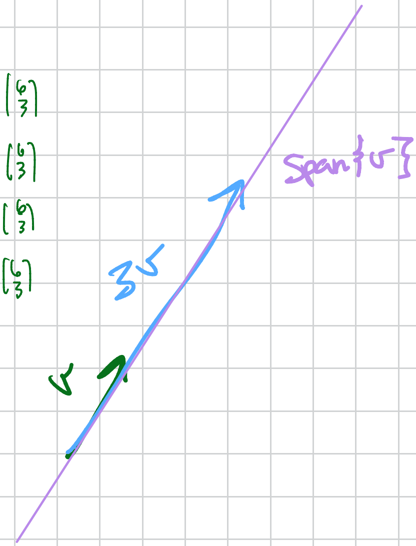
Q_0: what are some vectors in it?

$$x\begin{bmatrix}2\\1\end{bmatrix}+y\begin{bmatrix}6\\3\end{bmatrix}=x\begin{bmatrix}2\\1\end{bmatrix}+3y\begin{bmatrix}2\\1\end{bmatrix}$$
$$=(x+3y)\begin{bmatrix}2\\1\end{bmatrix}$$

$$\boxed{A:\ \text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}6\\3\end{bmatrix}\right\}=\text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}}$$

✓

$$\begin{bmatrix}3\\3/2\end{bmatrix}=\frac{3}{2}\begin{bmatrix}2\\1\end{bmatrix}+0\begin{bmatrix}6\\3\end{bmatrix}$$
$$\begin{bmatrix}30\\15\end{bmatrix}=3\begin{bmatrix}2\\1\end{bmatrix}+4\begin{bmatrix}6\\3\end{bmatrix}$$
$$\begin{bmatrix}2\\1\end{bmatrix}=1\begin{bmatrix}2\\1\end{bmatrix}+0\begin{bmatrix}6\\3\end{bmatrix}$$
$$\begin{bmatrix}-4\\-2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}-\begin{bmatrix}6\\3\end{bmatrix}$$



Example: Solve the system and write out the corresponding vector equation. How can the system be interpreted geometrically?

$$\begin{cases}x-y=8\\2x-2y=16\end{cases}$$
6x-y=3

$$\Rightarrow\begin{bmatrix}x-y\\2x-2y\end{bmatrix}=\begin{bmatrix}8\\16\end{bmatrix}\Rightarrow\begin{bmatrix}x\\2x\end{bmatrix}+\begin{bmatrix}-y\\-2y\end{bmatrix}=\begin{bmatrix}8\\16\end{bmatrix}\Rightarrow\boxed{x\begin{bmatrix}1\\2\end{bmatrix}+y\begin{bmatrix}-1\\-2\end{bmatrix}=\begin{bmatrix}8\\16\end{bmatrix}}$$

Q: consistent? yes!

Q: How many soln? infinitely many.

Example: What is Span{[1;-1;0],[2;7;0],[10;-6;0]}?

Q_1: $\mathrm{Span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}, \begin{bmatrix}10\\-6\\0\end{bmatrix}\right\}$ what is it?

Q_2: (you try it!) write down a few vectors in the span.

need to pick scalars x, y, z then compute

$$x\begin{bmatrix}1\\-1\\0\end{bmatrix}+y\begin{bmatrix}2\\7\\0\end{bmatrix}+z\begin{bmatrix}10\\-6\\0\end{bmatrix} = \begin{bmatrix}*\\*\\*\end{bmatrix}$$ tell me these.

$$\begin{bmatrix}13\\0\\0\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}, \begin{bmatrix}65\\21\\0\end{bmatrix}, \begin{bmatrix}21\\-13\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\-2\\0\end{bmatrix}, \begin{bmatrix}3\\-3\\0\end{bmatrix}, \dots, \begin{bmatrix}a\\b\\0\end{bmatrix}$$

is every vector $\begin{bmatrix}a\\b\\0\end{bmatrix}$ possible?

$$x\begin{bmatrix}1\\-1\\0\end{bmatrix}+y\begin{bmatrix}2\\7\\0\end{bmatrix}+z\begin{bmatrix}10\\-6\\0\end{bmatrix} = \begin{bmatrix}a\\b\\0\end{bmatrix}$$ $\boxed{\text{consistent?}}$ for any choice of a, b.

no pivot in aug column!!

$$\Rightarrow \left[\begin{array}{ccc|c}1 & 2 & 10 & a\\-1 & 7 & -6 & b\\0 & 0 & 0 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 2 & 10 & a\\0 & 9 & 4 & a+b\\0 & 0 & 0 & 0\end{array}\right]$$

yes - $\begin{bmatrix}a\\b\\0\end{bmatrix}$ is always in the span $\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}, \begin{bmatrix}10\\-6\\0\end{bmatrix}\right\}$.

Summary

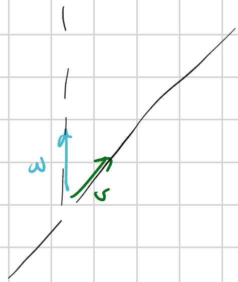
The whole lecture was about drawing pictures of systems of linear equations.

- **Points** and **vectors** are two ways of drawing elements of $\mathbf{R}^n$. Vectors are drawn as arrows.
- Vector addition, subtraction, and scalar multiplication have geometric interpretations.
- A **linear combination** is a sum of scalar multiples of vectors. This is also a geometric construction, which leads to lots of pretty pictures.
- The span of a set of vectors is the set of all linear combinations of those vectors. It is also fun to draw.
- A system of linear equations is equivalent to a vector equation, where the unknowns are the coefficients of a linear combination.

What does span look like?

① $\text{Span}\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}6\\3\end{bmatrix}\right\}$ looks like a line.

Same line as $\text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$.



② $\text{Span}\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$ looks like $\mathbb{R}^2$

③ $\text{Span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}, \begin{bmatrix}10\\-6\\0\end{bmatrix}\right\}$ looks like "The floor" a plane in $\mathbb{R}^3$.

§2.1

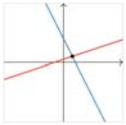
Chapter 2

Systems of Linear Equations: Geometry

Motivation

We want to think about the *algebra* in linear algebra (systems of equations and their solution sets) in terms of *geometry* (points, lines, planes, etc).

$$\begin{aligned} x - 3y &= -3 \\ 2x + y &= 8 \end{aligned} \quad \rightsquigarrow$$



Section 2.1

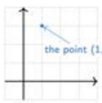
Vectors

Points and Vectors

We have been drawing elements of $\mathbf{R}^n$ as points in the line, plane, space, etc. We can also draw them as arrows.

Definition

A **point** is an element of $\mathbf{R}^n$, drawn as a point (a dot).



A **vector** is an element of $\mathbf{R}^n$, drawn as an arrow. When we think of an element of $\mathbf{R}^n$ as a vector, we'll usually write it vectically, like a matrix with one column:

$$v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

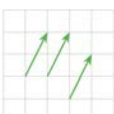
[interactive]

The difference is purely psychological: *points and vectors are just lists of numbers.*

Points and Vectors

So why make the distinction?

A vector need not start at the origin: *it can be located anywhere!* In other words, an arrow is determined by its length and its direction, not by its location.



These arrows all represent the vector $\binom{1}{2}$.

> However, unless otherwise specified, we'll assume a vector starts at the origin.

Vector Algebra

Definition

- We can add two vectors together:

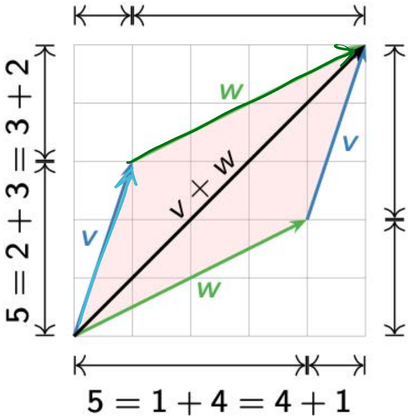
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix}.$$

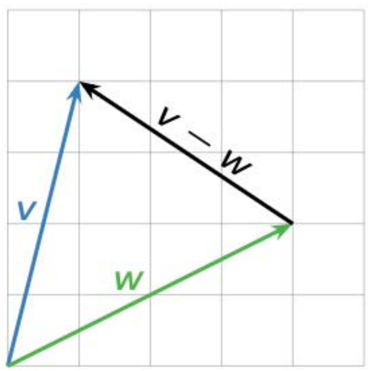
- We can multiply, or **scale**, a vector by a real number c:

$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \cdot x \\ c \cdot y \\ c \cdot z \end{pmatrix}.$$

We call c a **scalar** to distinguish it from a vector. If v is a vector and c is a scalar, cv is called a **scalar multiple** of v.

Vector Addition and Subtraction: Geometry

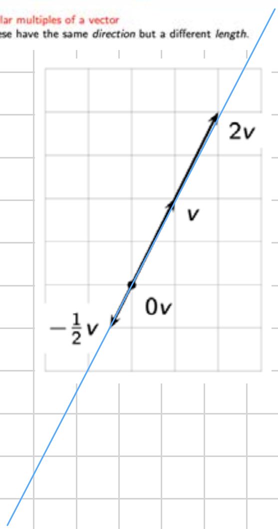




Scalar Multiplication: Geometry

Scalar multiples of a vector

These have the same *direction* but a different *length*.



Linear Combinations

We can add and scalar multiply in the same equation:

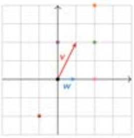
$$w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

where $c_1, c_2, \ldots, c_p$ are scalars, $v_1, v_2, \ldots, v_p$ are vectors in $\mathbf{R}^n$, and w is a vector in $\mathbf{R}^n$.

Definition

We call w a **linear combination** of the vectors $v_1, v_2, \ldots, v_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the **weights** or **coefficients**.

Example



Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

What are some linear combinations of v and w?

- $v + w$
- $v - w$
- $2v + 0w$
- $2w$
- $-v$

[interactive: 2 vectors] [interactive: 3 vectors]

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=1,2&v2=1,0&range=5&captions=combo&nomove=true&labels=v,w

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,1,-1&v3=-1,1,4&range=5&captions=combo&nomove=true

You Try It!

Example. If v=[1;2] and w=[1;0] then find

- v+w
- v-w
- 2v+0w
- 2w
- -v

Poll

Poll

Is there any vector in $\mathbf{R}^2$ that is *not* a linear combination of v and w?

Poll

Poll

Is there any vector in $\mathbf{R}^2$ that is *not* a linear combination of v and w?

More Examples

What are some linear combinations of $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

What are some linear combinations of $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

§2.2

Section 2.2

Vector Equations and Spans

Summary

The **vector equation**

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b,$$

where $v_1, v_2, \ldots, v_p, b$ are vectors in $\mathbf{R}^n$ and $x_1, x_2, \ldots, x_p$ are scalars, has the same solution set as the linear system with augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array}\right),$$

where the v_i's and b are the columns of the matrix.

Systems of Linear Equations

Solve the following system of linear equations:

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3. \end{aligned}$$

We can write all three equations at once as vectors:

$$\begin{pmatrix} x - y \\ 2x - 2y \\ 6x - y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

We can write this as a linear combination:

$$x \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

So we are asking:

Question: Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

Definition

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$. In symbols:

"the set of" "such that"

$$\boxed{\mathrm{Span}\{v_1, v_2, \ldots, v_p\} = \{x_1 v_1 + x_2 v_2 + \cdots + x_p v_p \mid x_1, x_2, \ldots, x_p \text{ in } \mathbf{R}\}.}$$

Synonyms: $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is the subset **spanned by** or **generated by** $v_1, v_2, \ldots, v_p$.

Now we have several equivalent ways of making the same statement:

1. A vector b is in the span of $v_1, v_2, \ldots, v_p$.
2. The vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$

has a solution.

3. The linear system with augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array}\right)$$

is consistent.

Example: Is the vector b in the span of v and w, where b=[1;2;1], v=[1;0;-1] and w=[0;1;1]?

Q: Is $b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ in the span of $v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ & $w = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$?

You Try It!

Poll

Poll

How many vectors are in Span $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$?

A. Zero
B. One
C. Infinity

Example: If v1=[2;1], what is Span{v1,3v1}?

Q_1: Span$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$ what is it?

Q_0: what are some vectors in it?

Example: Solve the system and write out the corresponding vector equation. How can the system be interpreted geometrically?

x-y=8
2x-2y=16
6x-y=3

Q: consistent?

Q: How many soln?

Example: What is Span{[1;-1;0],[2;7;0],[10;-6;0]}?

Q_1: $\text{Span}\left\{ \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}, \begin{bmatrix}10\\-6\\0\end{bmatrix} \right\}$ what is it?

Q_2: (you try it!) write down a few vectors in the span.

Need to pick scalars x, y, z then compute

$$x\begin{bmatrix}1\\-1\\0\end{bmatrix} + y\begin{bmatrix}2\\7\\0\end{bmatrix} + z\begin{bmatrix}10\\-6\\0\end{bmatrix} = \begin{bmatrix}*\\*\\*\end{bmatrix}$$

tell me these.

Summary

The whole lecture was about drawing pictures of systems of linear equations.

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What does span look like?

① $\text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$ looks like

② $\text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ looks like

③ $\text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ -6 \\ 0 \end{bmatrix} \right\}$ looks like