Chapter 2

Systems of Linear Equations: Geometry

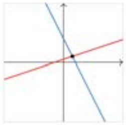
§2.1

Motivation

We want to think about the *algebra* in linear algebra (systems of equations and their solution sets) in terms of *geometry* (points, lines, planes, etc).

algebra

$$\begin{cases} x - 3y = -3 \\ 2x + y = 8 \end{cases} \rightsquigarrow$$



geometry.

$$x\begin{pmatrix}1\\2\end{pmatrix} + y\begin{pmatrix}-3\\1\end{pmatrix} = \begin{pmatrix}-3\\8\end{pmatrix}.$$

Date	Topic	Materials	WeBWorK	Quiz/Exam	Remarks
M Aug 24	Overview and 1.1, Systems of linear equations	1.1 Slides			
W Aug 26	1.1 (continued) and 1.2, Row reduction	1.2 Slides			
F Aug 28	Studio: 1.1 and 1.2	Worksheet (solutions)	Warmup	No quiz	
M Aug 31	1.2 (continued) and 1.3, Parametric form	More 1.2 and 1.3 Slides			
W Sep 2	1.3 (continued) and 2.1 and 2.2: Vectors, vector equations, and spans	2.1+2.2 Slides	1.1		
F Sep 4	Studio: 1.2 and 1.3	1.2-1.3 Worksheet (solutions) 1.2-1.3 Supplement (solutions)		Quiz 1: Syllabus	
M Sep 7			*Labor Day, No Class*		
W Sep 9	2.1 and 2.2, continued		1.2 and 1.3		
F Sep 11	Studio: 2.1, 2.2	2.1-2.2 Worksheet (solutions) 2.1-2.2 Supplement (solutions)		Quiz 2: 1.1-1.3	

Section 2.1

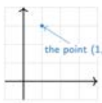
Vectors

Points and Vectors

We have been drawing elements of $\mathbf{R}^n$ as points in the line, plane, space, etc. We can also draw them as arrows.

Definition

A **point** is an element of $\mathbf{R}^n$, drawn as a point (a dot).



A **vector** is an element of $\mathbf{R}^n$, drawn as an arrow. When we think of an element of $\mathbf{R}^n$ as a vector, we'll usually write it vectically, like a matrix with one column:

$$v = \begin{pmatrix}1\\3\end{pmatrix}.$$

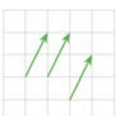
[interactive]

The difference is purely psychological: *points and vectors are just lists of numbers.*

Points and Vectors

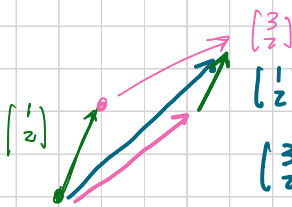
So why make the distinction?

A vector need not start at the origin: *it can be located anywhere!* In other words, an arrow is determined by its length and its direction, not by its location.



These arrows all represent the vector $\begin{pmatrix}1\\2\end{pmatrix}$.

However, unless otherwise specified, we'll assume a vector starts at the origin.



$$\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}4\\4\end{bmatrix}$$

$$\begin{bmatrix}3\\2\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}4\\4\end{bmatrix}.$$

Vector Algebra

Definition

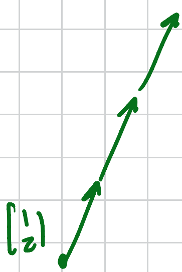
- We can add two vectors together:

$$\begin{pmatrix}a\\b\\c\end{pmatrix} + \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a+x\\b+y\\c+z\end{pmatrix}.$$

- We can multiply, or **scale**, a vector by a real number c:

$$c\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}c\cdot x\\c\cdot y\\c\cdot z\end{pmatrix}.$$

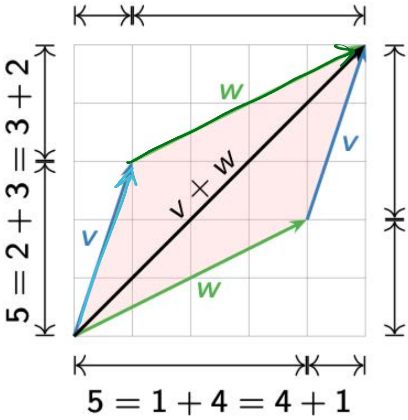
We call c a **scalar** to distinguish it from a vector. If v is a vector and c is a scalar, cv is called a **scalar multiple** of v.



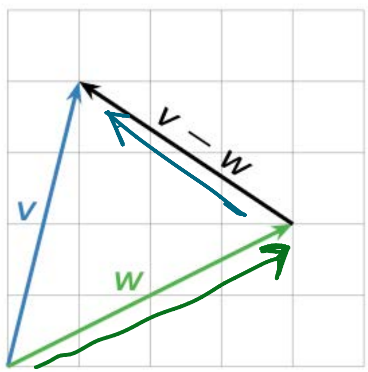
scalar

$$3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$$

Vector Addition and Subtraction: Geometry



parallelogram rule

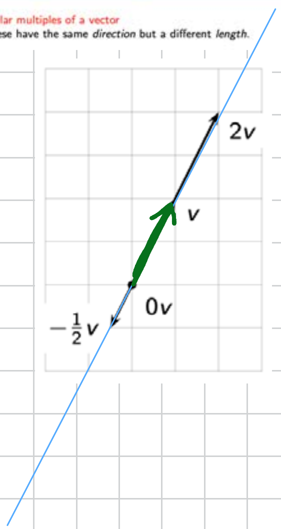


$w + (v - w) = v$

Scalar Multiplication: Geometry

Scalar multiples of a vector

These have the same direction but a different *length*.



$0\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$2\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$

$-\frac{1}{2}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-1/2\\-1\end{bmatrix}.$

Linear Combinations

We can add and scalar multiply in the same equation:

$$w = c_1v_1 + c_2v_2 + \cdots + c_pv_p$$

where $c_1, c_2, \ldots, c_p$ are scalars, $v_1, v_2, \ldots, v_p$ are vectors in $\mathbf{R}^n$, and w is a vector in $\mathbf{R}^n$.

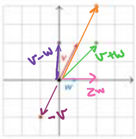
For some (existential quantifier "$\exists$" (exists))

w is a linear combination of $v_1, \ldots, v_p$.

Definition

We call w a **linear combination** of the vectors $v_1, v_2, \ldots, v_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the **weights** or **coefficients**.

Example



Let $v = \begin{pmatrix}1\\2\end{pmatrix}$ and $w = \begin{pmatrix}1\\0\end{pmatrix}$.

What are some linear combinations of v and w?

- $v + w$
- $v - w$
- $2v + 0w$
- $2w$
- $-v$

[interactive: 2 vectors] [interactive: 3 vectors]

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https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,1,-1&v3=-1,1,4&range=5&captions=combo&nomove=true

You Try It!

Example. If v=[1;2] and w=[1;0] then find

- v+w
- v-w
- 2v+0w
- 2w
- -v

You Try it

Poll

Poll

Is there any vector in $\mathbf{R}^2$ that is *not* a linear combination of v and w?

$\boxed{\text{yes}}$

If $\begin{bmatrix}a\\b\end{bmatrix}$ is any vector (randomly chosen) in $\mathbb{R}^2$

does there always exist scalars c_1 & c_2 such that

$$c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}?$$

rewrite this vector equation as a system of linear equation

$$\begin{bmatrix}c_1\\2c_1\end{bmatrix} + \begin{bmatrix}c_2\\0\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix} \Rightarrow \begin{cases} c_1 + c_2 = a \\ 2c_1 = b \end{cases}$$

$\Rightarrow$ need to solve system w/ augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 2 & 0 & b \end{array}\right] \sim \xrightarrow{-2R_1+R_2\to} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -2 & b-2a \end{array}\right]$$

no pivot in any col.

Original system is consistent! (unique soln b/c no free var col.)

Poll

Poll

Is there any vector in $\mathbf{R}^2$ that is *not* a linear combination of v and w?

More Examples

① What are some linear combinations of $v = \binom{2}{1}$?

② What are all linear combinations of

$v = \binom{2}{2}$ and $w = \binom{-1}{-1}$?

What are some linear combinations of $v = \binom{2}{1}$?

particular linear combinations of $\begin{bmatrix}2\\1\end{bmatrix}$.

arbitrary linear combination of $\begin{bmatrix}2\\1\end{bmatrix}$

$$\underbrace{\begin{bmatrix}4\\2\end{bmatrix}, \begin{bmatrix}-2\\-1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}3\\3/2\end{bmatrix}, \begin{bmatrix}-4\\-2\end{bmatrix}}, \quad c\begin{bmatrix}2\\1\end{bmatrix}$$

what shape do these make if I collect them all?

Ans. a line in the direction $\begin{bmatrix}2\\1\end{bmatrix}$.

What are all linear combinations of

$v = \binom{2}{2}$ and $w = \binom{-1}{-1}$?

$$c_1\begin{bmatrix}2\\2\end{bmatrix} + c_2\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}*\\*\end{bmatrix}$$ want to collect some of these.

$c_1=1$, $c_2=1$ get $\begin{bmatrix}1\\1\end{bmatrix}$ $\qquad$ $c_1=4$, $c_2=3$ $\begin{bmatrix}1\\1\end{bmatrix}$ $\qquad$ $c_1=8$, $c_2=7$ $\begin{bmatrix}9\\9\end{bmatrix}$,

$c_1=2$, $c_2=4$ get $\begin{bmatrix}0\\0\end{bmatrix}$ $\qquad$ $c_1=8$, $c_2=16$ $\begin{bmatrix}0\\0\end{bmatrix}$,

$$\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}9\\9\end{bmatrix}, \ldots$$

what shape do these make if I collect them all?

Ans. a line in the direction $\begin{bmatrix}1\\1\end{bmatrix}$.

§2.2

Section 2.2

Vector Equations and Spans

Systems of Linear Equations

Solve the following system of linear equations:

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3. \end{aligned}$$

We can write all three equations at once as vectors:

$$\begin{pmatrix} x - y \\ 2x - 2y \\ 6x - y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

$$\begin{bmatrix} x \\ 2x \\ 6x \end{bmatrix} + \begin{bmatrix} -y \\ -2y \\ -y \end{bmatrix} = x\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

We can write this as a linear combination:

$$\boxed{x\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.}$$

So we are asking:

Question: Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

#

$$\left[\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right]$$

Summary

The **vector equation**

$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = b,$$

where $v_1, v_2, \ldots, v_p, b$ are vectors in $\mathbf{R}^n$ and $x_1, x_2, \ldots, x_p$ are scalars, has the same solution set as the linear system with augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array}\right),$$

where the v_i's and b are the columns of the matrix.

This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

Definition

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted Span$\{v_1, v_2, \ldots, v_p\}$. In symbols:

"the set of" "such that"

$$\boxed{\text{Span}\{v_1, v_2, \ldots, v_p\} = \{x_1v_1 + x_2v_2 + \cdots + x_pv_p \mid x_1, x_2, \ldots, x_p \text{ in } \mathbf{R}\}.}$$

Synonyms: Span$\{v_1, v_2, \ldots, v_p\}$ is the subset **spanned by** or **generated by** $v_1, v_2, \ldots, v_p$.

the span of $v_1, \ldots, v_p$

is

(*) hard defn.

start

end

Now we have several equivalent ways of making the same statement:

1. A vector b is in the span of $v_1, v_2, \ldots, v_p$.
2. The vector equation
$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = b$$
has a solution.
3. The linear system with augmented matrix
$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array}\right)$$
is consistent.

Check

$$1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} ✓$$

Example: Is the vector b in the span of v and w, where b=[1;2;1], v=[1;0;-1] and w=[0;1;1]?

Q: Is $b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ in the span of $v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ & $w = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$?

don't need guess & check, need to row reduce something!

$$c_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{aligned} c_1 &= 1 \\ c_2 &= 2 \\ 0 &= 0 \end{aligned}$$

You Try It!

Poll

Poll

How many vectors are in Span$\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$?

A. Zero

B. One

C. Infinity

$1\begin{bmatrix}0\\0\\0\end{bmatrix}, 3\begin{bmatrix}0\\0\\0\end{bmatrix}, -1\begin{bmatrix}0\\0\\0\end{bmatrix}, 0\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ ✓

Example: If v1=[2;1], what is Span{v1,3v1}?

Q_1: span$\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}6\\3\end{bmatrix}\right\}$ what is it? (geometrically) a line in $\mathbb{R}^2$ ✓ (in the direction of $\begin{bmatrix}2\\1\end{bmatrix}$)

Q_0: what are some vectors in it?

$0\begin{bmatrix}2\\1\end{bmatrix}+0\begin{bmatrix}6\\3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ ✓

$1\begin{bmatrix}2\\1\end{bmatrix}+1\begin{bmatrix}6\\3\end{bmatrix}=\begin{bmatrix}8\\4\end{bmatrix}$ ✓

$1\begin{bmatrix}2\\1\end{bmatrix}-1\begin{bmatrix}6\\3\end{bmatrix}=\begin{bmatrix}-4\\-2\end{bmatrix}$ ✓

$3\begin{bmatrix}2\\1\end{bmatrix}+(-1)\begin{bmatrix}6\\3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

two different choices of scalars in the linear comb. get $\begin{bmatrix}0\\0\end{bmatrix}$ as an output.

all are scalar mult. of $\begin{bmatrix}2\\1\end{bmatrix}$.

$$\boxed{\text{Span}\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}6\\3\end{bmatrix}\right\}=\text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}}$$

Example: Solve the system and write out the corresponding vector equation. How can the system be interpreted geometrically?

x-y=8
2x-2y=16
6x-y=3

REF (not RREF)

$$\left[\begin{array}{cc|c}1&-1&8\\2&-2&16\\6&-1&3\end{array}\right]\sim_{-6R_1+R_3}\left[\begin{array}{cc|c}1&-1&8\\0&0&0\\0&5&-45\end{array}\right]\sim\left[\begin{array}{cc|c}1&-1&8\\0&1&-9\\0&0&0\end{array}\right]$$

$$\sim\left[\begin{array}{cc|c}1&0&-1\\0&1&-9\\0&0&0\end{array}\right]$$

$x=-1$
$y=-9$
$0=0$

So only soln is $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-1\\-9\end{bmatrix}$.

Q: consistent? yes

Q: How many soln? only one (unique soln). The solution is a point in $\mathbb{R}^2$

Example: What is Span{[1;-1;0],[2;7;0],[10;-6;0]}?

Q_1: $\text{Span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}2\\7\\0\end{bmatrix},\begin{bmatrix}10\\-6\\0\end{bmatrix}\right\}$ what is it? describe geometrically.

① a line in $\mathbb{R}^3$?
$\boxed{\text{② a plane in } \mathbb{R}^3 \text{?}}$ ✓
③ all of $\mathbb{R}^3$?

Q_2: (you try it!) write down a few vectors in the span.

Need to pick scalars x,y,z then compute

$$x\begin{bmatrix}1\\-1\\0\end{bmatrix}+y\begin{bmatrix}2\\7\\0\end{bmatrix}+z\begin{bmatrix}10\\-6\\0\end{bmatrix}=\begin{bmatrix}*\\*\\0\end{bmatrix}$$

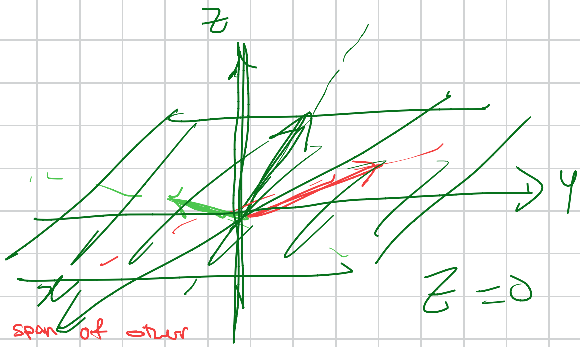
tell me these.

If $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\1\\-1\end{bmatrix}$ Then the vector in the span is $\begin{bmatrix}-5\\10\\0\end{bmatrix}$

$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$ " —— " is $\begin{bmatrix}13\\0\\0\end{bmatrix}$

$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}$ " —— " is $\begin{bmatrix}2\\7\\0\end{bmatrix}$

3 different directions so so the span of $\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}2\\7\\0\end{bmatrix},\begin{bmatrix}10\\-6\\0\end{bmatrix}$, ...



Summary

The whole lecture was about drawing pictures of systems of linear equations.

- **Points** and **vectors** are two ways of drawing elements of $\mathbf{R}^n$. Vectors are drawn as arrows.
- Vector addition, subtraction, and scalar multiplication have geometric interpretations.
- A **linear combination** is a sum of scalar multiples of vectors. This is also a geometric construction, which leads to lots of pretty pictures.
- The span of a set of vectors is the set of all linear combinations of those vectors. It is also fun to draw.
- $\boxed{\text{A system of linear equations is equivalent to a vector equation, where the unknowns are the coefficients of a linear combination.}}$

↑ system of linear eqns vs. one vector in span of other

Example: Solve the system and write out the corresponding vector equation. How can the system be interpreted geometrically?

$$\begin{cases} x-y=8\\ 2x-2y=16\end{cases} \Rightarrow \begin{bmatrix}x-y\\2x-2y\end{bmatrix}=\begin{bmatrix}8\\16\end{bmatrix} \Rightarrow \begin{bmatrix}x\\2x\end{bmatrix}+\begin{bmatrix}-y\\-2y\end{bmatrix}=\begin{bmatrix}8\\16\end{bmatrix}$$

↑ Solving this system.

$$\Rightarrow x\begin{bmatrix}1\\2\end{bmatrix}+y\begin{bmatrix}-1\\-2\end{bmatrix}=\begin{bmatrix}8\\16\end{bmatrix}.$$

same as asking for x,y scalars to get $\begin{bmatrix}8\\16\end{bmatrix}$ as a linear combo of $\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\-2\end{bmatrix}$

Example: Solve the system and write out the corresponding vector equation. How can the system be interpreted geometrically?

$$\begin{cases} x-y=8 \\ 2x-2y=16 \end{cases} \Rightarrow \begin{bmatrix} x-y \\ 2x-2y \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ 2x \end{bmatrix} + \begin{bmatrix} -y \\ -2y \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \end{bmatrix}$$

Solving this system.

$$\Rightarrow \boxed{x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \end{bmatrix}}.$$

same as asking for x, y scalars to get $\begin{bmatrix} 8 \\ 16 \end{bmatrix}$ as a linear comb of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

Q_1: What are all the solutions?

Q_2: What are some solutions?

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 \\ -8 \end{bmatrix}, \begin{bmatrix} 9 \\ 1 \end{bmatrix}, \begin{bmatrix} 10 \\ 2 \end{bmatrix}, \begin{bmatrix} 11 \\ 3 \end{bmatrix}, \begin{bmatrix} 16 \\ 8 \end{bmatrix}.$$

$x = 8+y$

how to interpret these w/ vectors.

x pivot, y (free)

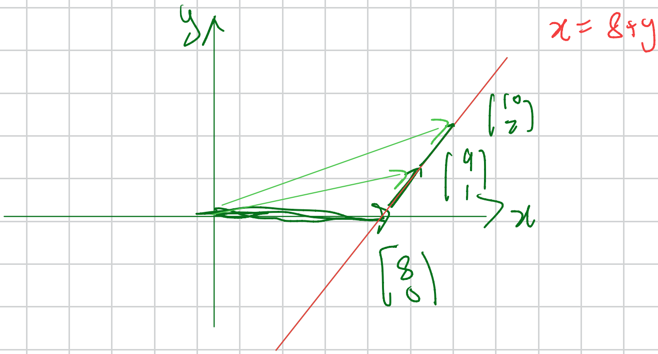
$$\left[\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 8 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x-y=8 \\ y=y \text{ (free)} \end{cases} \Rightarrow \boxed{\begin{cases} x=8+y \\ y=y \text{ (free)} \end{cases}}$$

parametric eqn form.

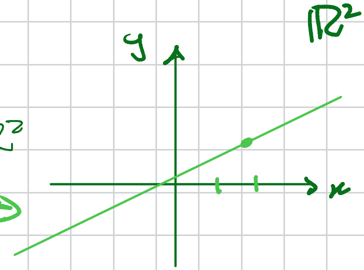
So a soln (in vector form)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8+y \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix} + \begin{bmatrix} y \\ y \end{bmatrix} = \boxed{\begin{bmatrix} 8 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \end{bmatrix}}.$$



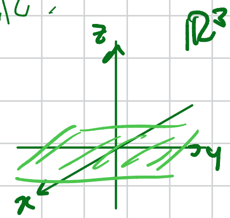
What does span look like?

① $\text{Span}\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}6\\3\end{bmatrix}\right\}$ looks like a line in $\mathbb{R}^2$



② $\text{Span}\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right\}$ looks like ① line in $\mathbb{R}^2$?
② all of $\mathbb{R}^2$?

③ $\text{Span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}2\\7\\0\end{bmatrix},\begin{bmatrix}10\\-6\\0\end{bmatrix}\right\}$ looks like a plane in $\mathbb{R}^3$.



What is in the span?

$$x\begin{bmatrix}2\\1\end{bmatrix}+y\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$$

what can these be?

Solve by row reducing

$$\left[\begin{array}{cc|c}2&1&a\\1&0&b\end{array}\right]\sim\left[\begin{array}{cc|c}1&0&b\\2&1&a\end{array}\right]$$

$$\underset{-2R_1+R_2}{\sim}\left[\begin{array}{cc|c}1&0&b\\0&1&a-2b\end{array}\right]$$

Q: Is this system consistent no matter how $\begin{bmatrix}a\\b\end{bmatrix}$ is chosen?

A: yes. b/c there is always a choice for x, y that work

pun chline never a pivot in augmented column.

$\text{Span}\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right\}=\mathbb{R}^2$ b/c $\begin{bmatrix}a\\b\end{bmatrix}$ is a lin comb of $\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}$ for any a,b