

Definition:
The product of A with a vector x in $\mathbb{R}^n$ is the linear combination

$$Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \left(\text{also means } \sum_{i=1}^n x_i v_i \right)$$

The output is a vector in $\mathbb{R}^m$.

(Note: In the original image, a red box highlights the scalar product $x_i v_i$ and a red arrow points to the text "these must be equal".)

Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 32 \\ 49 \end{pmatrix}$$

Ans. $\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ ^{matrix} times $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$ ^{vector} is a linear combination of the columns of the matrix w/ scalars from the vector.

Matrix Equations

Let v_1, v_2, v_3 be vectors in $\mathbb{R}^3$. How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication? **linear comb.**

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}$$

Matrix Equations

Let $v_1, v_2, \dots, v_n$ and b be vectors in $\mathbb{R}^m$. Consider the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_nv_n = \vec{b}.$$

It is equivalent to the matrix equation

$$Ax = b$$

where

Has m rows $\times$ n columns.

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

X has n entries
 b has m entries.

$m = \# \text{ rows of } A$ $b \text{ in } \mathbb{R}^m$
 $n = \# \text{ cols of } A$ $X \text{ in } \mathbb{R}^n$

Conversely, if A is any $m \times n$ matrix, then

$$Ax = b$$

is equivalent to the
vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = \vec{b}$$

where $v_1, \dots, v_n$ are the columns of A , and $x_1, \dots, x_n$ are the entries of x .

Linear Systems, Vector Equations, Matrix Equations, ...

We now have four equivalent ways of writing (and thinking about) linear systems:

1. As a system of equations:

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5 \end{aligned}$$

Solution: $x_1 = 2, x_2 = 1$ that works.

2. As an augmented matrix:

$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array} \right)$$

3. As a vector equation $x_1v_1 + \dots + x_nv_n = b$:

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a matrix equation $Ax = b$:

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

In particular, all four have the same solution set.

We will move back and forth freely between these over and over again for the rest of the semester. Get comfortable with them now!

Very Important Fact That Will Appear on Every Midterm and the Final

$Ax = b$ has a solution

$\iff$ there exist $x_1, \dots, x_n$ such that $A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = b$

"if and only if"

$\iff$ there exist $x_1, \dots, x_n$ such that $x_1v_1 + \dots + x_nv_n = b$

$\iff b$ is a linear combination of $v_1, \dots, v_n$

$\iff b$ is in the span of the columns of A .

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$

Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?

well, convert matrix.

matrix equation $Ax = b$.

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

convert to augmented matrix.

REF ✓

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & -1 & 2 \end{array} \right] \sim \left[\begin{array}{cc|c} -1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & -1 & 2 \end{array} \right] \sim \begin{matrix} -R_1 \rightarrow \\ R_2 + R_1 \rightarrow \\ R_3 + R_1 \rightarrow \end{matrix} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & 4 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 8 \end{array} \right]$$

Q: does $Ax = b$ have a soln?

A: No, The system is inconsistent b/c pivot in aug. col. of aug matrix $[A|b]$.

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$

Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?

When Solutions Always Exist

Here are criteria for a linear system to always have a solution.

Theorem

Let A be an $m \times n$ (non-augmented) matrix. The following are equivalent:

1. $Ax = b$ has a solution for all b in $\mathbb{R}^m$.
2. The span of the columns of A is all of $\mathbb{R}^m$.
3. A has a pivot in each row.

Important! This means that for given A , either they're all true, or they're all false.

non-example of these ↑

$$[A|b] \sim \dots \sim \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array} \right]$$

↑ RREF of A .

T/F A has a pivot in every row.

$$[A|b] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & -1 & 2 \end{array} \right]$$

The span of $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ is NOT $\mathbb{R}^3$.

you will do these already (naturally).

Properties of the Matrix-Vector Product

Let c be a scalar, u, v be vectors, and A a matrix.

- $A(u+v) = Au + Av$
- $A(cu) = cAu$

Consequence: If u and v are solutions to $Ax = 0$, then so is every vector in $\text{Span}(u, v)$. Why?

Important

The set of solutions to $Ax = 0$ is a span.

$$\checkmark \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\checkmark \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}) = 2 \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$Ax = b$$

Summary

- We have four equivalent ways of writing a system of linear equations:
 1. As a system of equations.
 2. As an augmented matrix.
 3. As a vector equation.
 4. As a matrix equation $Ax = b$.
- $Ax = b$ is consistent if and only if b is in the span of the columns of A . The latter condition is geometric: you can draw pictures of it.
- $Ax = b$ is consistent for all b in $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.

Example where conditions ARE satisfied

<https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,0,-1&v3=-1,2,2&range=5&capopt=matrix>

Example where conditions are NOT satisfied

<https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,0,-1&v3=1,-1,2&range=5&capopt=matrix>

Section 2.4

Solution Sets

Plan For Today

Today we will learn to describe and draw the solution set of an arbitrary system of linear equations $Ax = b$, using space.

Recall: the solution set is the collection of all vectors x such that $Ax = b$ is true.

The solutions to $Ax = b$ could be
 * a line in $\mathbb{R}^3$
 * a plane in $\mathbb{R}^4$
 * a point in $\mathbb{R}^5$

Homogeneous Systems

Definition

A system of linear equations of the form $Ax = 0$ is called **homogeneous**.

Definition

A system of linear equations of the form $Ax = b$ with $b \neq 0$ is called **inhomogeneous**.

A homogeneous system always has the solution $x = 0$. This is called the **trivial solution**. The nonzero solutions are called **nontrivial**.

Observation

$Ax = 0$ has a nontrivial solution
 $\iff$ there is a free variable
 $\iff A$ has a column with no pivot.

$$\begin{cases} x_1 - x_2 = 0 \\ x_1 + x_3 = 0 \end{cases}$$

Q: are there any solutions to this homogeneous system?

A: If $x_1 = 0$ and $x_2 = 0$ then yes this is a solution.

Q: are there any other (non-obvious) solutions?

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right]$$

Pivot in every coefficient column
 $\Rightarrow$ **unique soln**

$$\text{So } A = \underline{I_2}$$

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}?$$

Observation

Since the last column (everything to the right of the $=$) was zero to begin, it will always stay zero! So it's not really necessary to write augmented matrices in the homogeneous case.

Q: are there other solutions to $Ax = 0$ other than $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$?

If yes, find them all (using parametric eqn/vector form)

Sols are a point in $\mathbb{R}^3$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & -7 & -6 & 0 \\ 0 & -3 & -3 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -7 & -6 & 0 \end{array} \right]$$

REF

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Rewrite as
 $x_1 = 0$
 $x_2 = 0$
 $x_3 = 0$

So only soln is
 $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Homogeneous Systems

Example

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$$

$b=0$
(homogeneous)

$$\left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 1 & -6 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

do RREF
↓
 $\left(\begin{array}{cc|c} 1 & -3 & 0 \\ 2 & -6 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array} \right)$

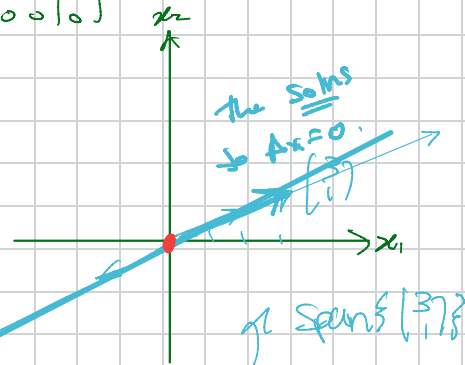
$$\begin{cases} x_1 - 3x_2 = 0 \\ x_2 = x_2 \text{ (free)} \end{cases}$$

$$\begin{cases} x_1 = 3x_2 \\ x_2 = x_2 \text{ (free)} \end{cases}$$

parametric eqn for
of solns
(so many)

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \text{ } x_2 \text{ any real number}$$

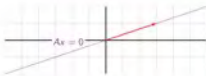
parametric vector form.



The solutions to $Ax=0$ are the span of $\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$.
Which is a line in $\mathbb{R}^2$.

Exam 1 on
Wed 9/23 @
6:30 pm

see Canvas for
locations.



Note: one free variable means the solution set is a line in $\mathbb{R}^2$ ($2 = \#$ variables $= \#$ columns).

<https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?lock=true&x=3,1&mat=1,-3;2,-6&range2=5>

Homogeneous Systems

Example

Question
What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

$b=0$
(homogeneous)

One solution is the
trivial solution

$$\cancel{x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}} \text{ or } x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ??$$

✓

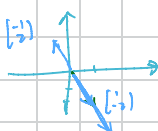
$$[A|0] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ -2 & 2 & -4 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} \textcircled{1} & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

RREF

$$\text{Span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} \right\}$$

is a line in $\mathbb{R}^2$



parametric eqn form

$$\begin{cases} x_1 - x_2 + 2x_3 = 0 \\ x_2 = x_2 \text{ (free)} \\ x_3 = x_3 \text{ (free)} \end{cases}$$

$$\begin{cases} x_1 = x_2 - 2x_3 \\ x_2 = x_2 \text{ (free)} \\ x_3 = x_3 \text{ (free)} \end{cases}$$

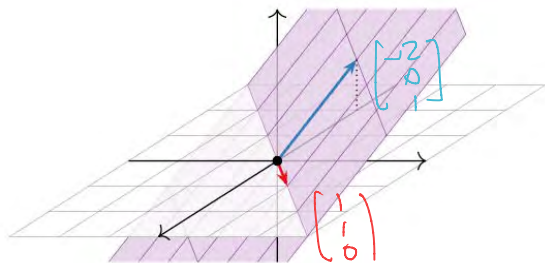
$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 - 2x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -2x_3 \\ 0 \\ x_3 \end{pmatrix}$$

$$X = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

parametric vector form.

Q: what is it? $A =$ a plane in $\mathbb{R}^3$,
Q: which plane? $\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

<https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?lock=true&x=0,0,0>



Note: two free variables means the solution set is a plane in $\mathbb{R}^3$ (3 = # variables = # columns).

$$\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a plane in $\mathbb{R}^3$

$$\begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Question

What is the solution set of $Ax = 0$, where $A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$

$b=0$ (homogeneous)

REF

$$(A|0) = \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ -2 & -3 & 4 & 5 & 0 \\ 2 & 4 & 0 & -2 & 0 \end{array} \right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & -1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -8 & -7 & 0 \\ 0 & 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{cases} x_1 = 8x_3 + 7x_4 \\ x_2 = -4x_3 - 3x_4 \\ x_3 = x_3 \text{ (free)} \\ x_4 = x_4 \text{ (free)} \end{cases}$$

$$X = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

parametric vector form

Q: What is the soln set?

A: Plane in $\mathbb{R}^4$

Q: Which plane?

A: Span $\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

Note: two free variables means the solution set is a plane in $\mathbb{R}^4$ (2 = # variables - # columns).

Parametric Vector Form

homogeneous system

Let A be an $m \times n$ matrix. Suppose that the free variables in the homogeneous equation $Ax = 0$ are, for example, x_1, x_2 , and x_3 .

1. Find the reduced row echelon form of A . (REF)
2. Write the parametric form of the solution set, including the redundant equations $x_1 = x_1, x_2 = x_2$, and $x_3 = x_3$. Put equations for all of the x_i in order. write free vars too
3. Make a single vector equation from these equations by putting x_1, x_2 , and x_3 as coefficients of vectors v_1, v_2 , and v_3 , respectively.

The solutions to $Ax = 0$ will then be expressed in the form

$$x = x_1 v_1 + x_2 v_2 + x_3 v_3$$

for some vectors v_1, v_2, v_3 in $\mathbb{R}^n$, and any scalars x_1, x_2, x_3 .

In this case, the solution set to $Ax = 0$ is

$$\text{Span}\{v_1, v_2, v_3\}.$$

Inhomogeneous Systems

Example

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \text{ and } b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$$

$b \neq 0$

(inhomogeneous)

<https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?x=-3,0&mat=1,-3,2,-6&lock=true&closed=true>

$$[A|b] = \left[\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & -6 & -6 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 0 & 0 \end{array} \right] \quad \begin{cases} x_1 - 3x_2 = -3 \\ x_2 = x_2 \text{ (free)} \end{cases}$$

$$\begin{cases} x_1 = -3 + 3x_2 \\ x_2 = x_2 \text{ (free)} \end{cases}$$

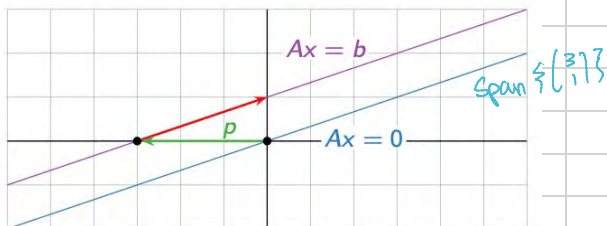
parametric eqn form

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 + 3x_2 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ x_2 \end{pmatrix}$$

$$X = \begin{pmatrix} -3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

parametric vector form



$P = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ is sometimes called a "particular soln to $Ax=b$ "

plus $x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ the "general soln to $Ax=0$ "

Q: What is it?

A: line in $\mathbb{R}^2$ but X doesn't pass through origin & parallel to $\text{span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$ the solns to $Ax=0$.
* but not a span.

Inhomogeneous Systems

Example

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$b \neq 0$

(inhomogeneous)

$$X = X_p + X_h$$

$\nwarrow$ particular soln to $Ax=b$
 $\swarrow$ general soln to $Ax=0$

$$[A|b] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ -2 & 2 & -4 & -2 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

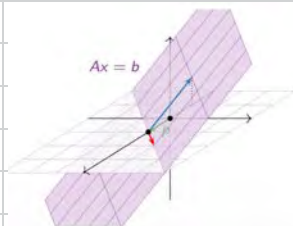
x_2, x_3 are free

$$\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ x_2 = x_2 \\ x_3 = x_3 \end{cases}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 + x_2 - 2x_3 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\downarrow \begin{cases} x_1 = 1 + x_2 - 2x_3 \\ x_2 = x_2 \\ x_3 = x_3 \end{cases} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$\uparrow$ X_p X_h



Homogeneous vs. Inhomogeneous Systems

Key Observation

The set of solutions to $Ax = b$, if it is nonempty, is obtained by taking one **specific or particular solution** p to $Ax = b$, and adding all solutions to $Ax = 0$.

Very Important

Let A be an $m \times n$ matrix. There are now two completely different things you know how to describe using spans:

- The **solution set**: for fixed b , this is all x such that $Ax = b$.
 - This is a span if $b = 0$, or a translate of a span in general (if it's consistent).
 - Lives in $\mathbb{R}^n$.
 - Computed by finding the parametric vector form.
- The **span of the columns**: this is all b such that $Ax = b$ is consistent.
 - This is the span of the columns of A .
 - Lives in $\mathbb{R}^m$.

* Shape is determined by # of free vars.
(line, plane, pt).

Summary

- The solution set to a **homogeneous** system $Ax = 0$ is a span: it always contains the **trivial solution** $x = 0$.
- The solution set to a **nonhomogeneous** system $Ax = b$ is either empty, or it is a translate of a span: namely, it is a translate of the solution set of $Ax = 0$.
- The solution set to $Ax = b$ can be expressed as a translate of a span by computing the **parametric vector form** of the solution.
- The solution set to $Ax = b$ and the span of the columns of A (from the previous lecture) are two completely different things, and you have to understand them separately.

Section 2.5

Linear Independence

Motivation

Sometimes the span of a set of vectors is "smaller" than you expect from the number of vectors.



Linear Independence

Definition

A set of vectors $\{v_1, v_2, \dots, v_p\}$ in $\mathbb{R}^n$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \dots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \dots = x_p = 0$. The set $\{v_1, v_2, \dots, v_p\}$ is **linearly dependent** otherwise.

In other words, $\{v_1, v_2, \dots, v_p\}$ is linearly dependent if there exist numbers $x_1, x_2, \dots, x_p$, not all equal to zero, such that

$$x_1 v_1 + x_2 v_2 + \dots + x_p v_p = 0.$$

This is called a **linear dependence relation** or an **equation of linear dependence**.

Like span, linear (in)dependence is another one of those big vocabulary words that you absolutely need to learn. Much of the rest of the course will be built on these concepts, and you need to know exactly what they mean in order to be able to answer questions on quizzes and exams (and solve real-world problems later on).

Note that linear (in)dependence is a notion that applies to a collection of vectors, not to a single vector, or to one vector in the presence of some others.

Warning: It is not meaningful to say "v is dependent on w", but you can say that the set $\{v, w\}$ is linearly dependent.

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} ?$$

a line in $\mathbb{R}^3$ ✓

$$(1) \text{Span} \{[1;1], [2;2], [0;0]\}$$

$$(2) \text{Span} \{[1;0;1], [0;1;0], [1;1;1]\}$$

$$\text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ is a plane in } \mathbb{R}^3$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Linear Independence and Matrix Columns

In general, $\{v_1, v_2, \dots, v_p\}$ is linearly independent if and only if the vector equation

$$x_1 v_1 + x_2 v_2 + \dots + x_p v_p = 0$$

has only the trivial solution, if and only if the matrix equation

$$Ax = 0$$

has only the trivial solution, where A is the matrix with columns $v_1, v_2, \dots, v_p$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_p \\ | & | & & | \end{pmatrix}$$

This is true if and only if the matrix A has a pivot in each column.

Important

- The vectors $v_1, v_2, \dots, v_p$ are linearly independent if and only if the matrix with columns $v_1, v_2, \dots, v_p$ has a pivot in each column.
- Solving the matrix equation $Ax = 0$ will either verify that the columns $v_1, v_2, \dots, v_p$ of A are linearly independent, or will produce a linear dependence relation.

Example: Are the vectors $v_1=[1;1;0]$, $v_2=[2;0;4]$, and $v_3=[1;2;-2]$ linearly dependent? Can you write v_3 as a vector in $\text{Span}\{v_1, v_2\}$?

Linear Independence

Critérios

Suppose that one of the vectors $\{v_1, v_2, \dots, v_p\}$ is a linear combination of the other ones (that is, it is in the span of the other ones):

$$v_1 = 2v_2 - \frac{1}{2}v_3$$

Theorem

A set of vectors $\{v_1, v_2, \dots, v_p\}$ is linearly dependent if and only if one of the vectors is in the span of the other ones.

Equivalently:

Theorem

A set of vectors $\{v_1, v_2, \dots, v_p\}$ is linearly dependent if and only if you can remove one of the vectors without shrinking the span.

Translation

A set of vectors is linearly independent if and only if, every time you add another vector to the set, the span gets bigger.

Linear Independence

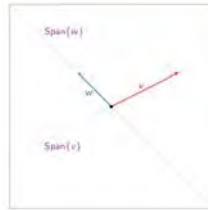
Pictures in $\mathbb{R}^2$



One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Linear Independence

Pictures in $\mathbb{R}^2$

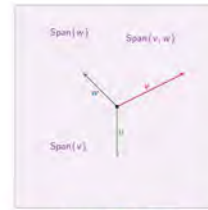


One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent
 • Neither is in the span of the other.
 • Span got bigger.

Linear Independence

Pictures in $\mathbb{R}^2$



One vector $\{v\}$:
Linearly independent if $v \neq 0$.

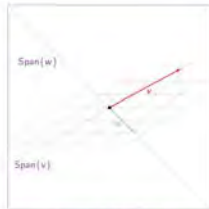
Two vectors $\{v, w\}$:
Linearly independent
 • Neither is in the span of the other.
 • Span got bigger.

Three vectors $\{v, w, u\}$:
Linearly dependent:
 • u is in $\text{Span}\{v, w\}$.
 • Span didn't get bigger after adding u .
 • Can remove u without shrinking the span.
 Also v is in $\text{Span}\{u, w\}$ and w is in $\text{Span}\{u, v\}$.

[interactive 2D: 2 vectors]
[interactive 2D: 3 vectors]

Linear Independence

Pictures in $\mathbb{R}^3$

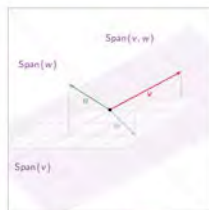


Two vectors $\{v, w\}$:
Linearly independent:
 • Neither is in the span of the other.
 • Span got bigger when we added w .

[interactive 3D: 2 vectors]
[interactive 3D: 3 vectors]

Linear Independence

Pictures in $\mathbb{R}^3$



Three vectors $\{v, w, u\}$:
Linearly independent: span got bigger when we added u .

[interactive 3D: 2 vectors]
[interactive 3D: 3 vectors]

<https://textbooks.math.gatech.edu/ila/demos/spans.html?captions=indep&v1=2,-1,1&v2=1,0,-1&v3=.5,-.5,1&bels=v,w,x&range=5>

Linear Dependence and Free Variables

Theorem

Let $v_1, v_2, \dots, v_p$ be vectors in $\mathbb{R}^n$, and consider the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_p \\ | & | & & | \end{pmatrix}$$

Then you can delete the columns of A without pivots (the columns corresponding to free variables), without changing $\text{Span}\{v_1, v_2, \dots, v_p\}$. The pivot columns are linearly independent, so you can't delete any more columns.

This means that each time you add a pivot column, then the span increases.

Example: Which of the vectors in the set $\{v_1, v_2, v_3, v_4, v_5\}$ can be removed without changing the span of the vectors?

$v_1 = [1; 0; 0]$, $v_2 = [1; 1; 0]$, $v_3 = [2; 2; 0]$, $v_4 = [1; 3; 0]$, $v_5 = [2; 2; 3]$

Linear Independence

Two more facts

Fact 1: Say $v_1, v_2, \dots, v_n$ are in $\mathbb{R}^m$. If $n > m$ then $\{v_1, v_2, \dots, v_n\}$ is linearly dependent: the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix}$$

cannot have a pivot in each column (it is too wide).

This says you can't have 4 linearly independent vectors in $\mathbb{R}^3$, for instance.

A wide matrix can't have linearly independent columns.

Example: Is $\{v_1, v_2, v_3\}$ linearly independent or linearly dependent? If the set is linearly dependent, then find a dependence relation on v_1, v_2, v_3 .

$$v_1 = [1; 1], v_2 = [0; 1], v_3 = [2; 3]$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$[A|0] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 3 & 0 \end{array} \right] \quad (\text{columns are } v_1, v_2, v_3) \quad A \text{ is } \underline{\text{wide}}$$

Fact 2: If one of $v_1, v_2, \dots, v_n$ is zero, then $\{v_1, v_2, \dots, v_n\}$ is linearly dependent. For instance, if $v_1 = 0$, then

$$1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + \dots + 0 \cdot v_n = 0$$

is a linear dependence relation.

A set containing the zero vector is linearly dependent.

Example: Is $\{v_1, v_2, v_3\}$ linearly independent or linearly dependent? If the set is linearly dependent, then find a dependence relation on v_1, v_2, v_3 .

$$v_1 = [1; 1; 0], v_2 = [0; 0; 0], v_3 = [0; 1; 1]$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim$$

Summary

- A set of vectors is **linearly independent** if removing one of the vectors shrinks the span; otherwise it's **linearly dependent**.
- There are several other criteria for linear (in)dependence which lead to pretty pictures.
- The columns of a matrix are linearly independent if and only if the RREF of the matrix has a pivot in every column.
- The pivot columns of a matrix A are linearly independent, and you can delete the non-pivot columns (the "free" columns) without changing the span of the columns.
- Wide matrices cannot have linearly independent columns.

Warning

These are not the official definitions!

Linear Independence

Definition

A set of vectors $\{v_1, v_2, \dots, v_p\}$ in $\mathbb{R}^n$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \dots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \dots = x_p = 0$. The set $\{v_1, v_2, \dots, v_p\}$ is **linearly dependent** otherwise.

This is the official textbook definition.