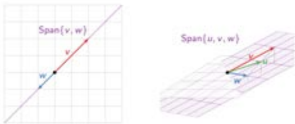
Section 2.5

Linear Independence

Motivation

Sometimes the span of a set of vectors is "smaller" than you expect from the number of vectors.



Textbook definitions.

Linear Independence

Definition
A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation
$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$$
has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$. The set $\{v_1, v_2, \ldots, v_p\}$ is **linearly dependent** otherwise.

Like span, linear (in)dependence is another one of those big vocabulary words that you absolutely need to learn. Much of the rest of the course will be built on these concepts, and you need to know exactly what they mean in order to be able to answer questions on quizzes and exams (and solve real-world problems later on).

In other words, $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if there exist numbers $x_1, x_2, \ldots, x_p$, not all equal to zero, such that
$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0.$$
This is called a **linear dependence relation** or an **equation of linear dependence**.

✓

$$(2)\begin{bmatrix}1\\1\end{bmatrix} + (-1)\begin{bmatrix}2\\2\end{bmatrix} + (0)\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$(0)\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}2\\2\end{bmatrix} + 15\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Note that linear (in)dependence is a notion that applies to a *collection of vectors*, not to a single vector, or to one vector in the presence of some others.

Example: what is

(1) Span{[1;1],[2;2],[0;0]}

$$\begin{bmatrix}1 & 2 & 0\\1 & 2 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 0\\0 & 0 & 0\end{bmatrix}$$

$$\text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\} ?$$

a line in $\mathbb{R}^2$ ✓

(2) Span{[1;0;1],[0;1;0],[1;1;1]}

$$\text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$$

is a plane in $\mathbb{R}^3$

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$$

Exam 1 This week
on Wednesday 9/23
@ 6:30 pm.
See Canvas for common exam locations

Checking Linear Independence

Question: Is $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}3\\1\\4\end{pmatrix}\right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation
$$x\begin{pmatrix}1\\1\\1\end{pmatrix} + y\begin{pmatrix}1\\-1\\2\end{pmatrix} + z\begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Warning: It is not meaningful to say "v is dependent on w", but you can say that the set {v,w} is linearly dependent.

Strategy: count pivots/free of the system w/ aug matrix $[A|0]$.

$$\left[\begin{array}{ccc|c}1 & 1 & 3 & 0\\1 & -1 & 1 & 0\\1 & 2 & 4 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 1 & 3 & 0\\0 & -2 & -2 & 0\\0 & 1 & 1 & 0\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1 & 0 & 2 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right] \quad \text{RREF}$$

z free

Coeff matrix A did NOT have a pivot in every col.

$x = -2z$
$y = -z$
$z = z$ (free)

A: No, the set $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}3\\1\\4\end{bmatrix}\right\}$ is linearly dependent.

Q2: Find a dependence relation?

A: one nontrivial soln is $z=1$

get $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-2\\-1\\1\end{bmatrix}$ so dep. rel. is

$$-2\begin{bmatrix}1\\1\\1\end{bmatrix} - 1\begin{bmatrix}1\\-1\\2\end{bmatrix} + 1\begin{bmatrix}3\\1\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Linear Independence and Matrix Columns

In general, $\{v_1, v_2, \dots, v_p\}$ is linearly independent if and only if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution, if and only if the matrix equation

$$Ax = 0$$

has only the trivial solution, where A is the matrix with columns $v_1, v_2, \dots, v_p$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

This is true if and only if the matrix A has a pivot in each column.

Important

- The vectors $v_1, v_2, \dots, v_p$ are linearly independent if and only if the matrix with columns $v_1, v_2, \dots, v_p$ has a pivot in each column.
- Solving the matrix equation $Ax = 0$ will either verify that the columns $v_1, v_2, \dots, v_p$ of A are linearly independent, or will produce a linear dependence relation.

Example: Are the vectors v1=[1;1;0], v2=[2;0;4], and v3=[1;2;-2] linearly dependent? Can you write v3 as a vector in Span{v1,v2}?

Q: $\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\4\end{bmatrix}, \begin{bmatrix}1\\2\\-2\end{bmatrix} \right\}$ linearly independent / $\boxed{\text{dependent}}$?

Soln $A = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 4 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 4 & -2 \\ 1 & 2 & 1 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 4 & -2 \\ 0 & 2 & -1 \end{array}\right]$

x_3 free

$\sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right]$

A: linearly dependent

b/c $A = [v_1\ v_2\ v_3]$ does not have a pivot in every column.

So -

$$2\begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}2\\0\\4\end{bmatrix} = \begin{bmatrix}1\\2\\-2\end{bmatrix}$$

and this gives the dependence relation

$$2\begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}2\\0\\4\end{bmatrix} - \begin{bmatrix}1\\2\\-2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Q_2: Is v_3 in Span$\{v_1, v_2\}$

Soln. $x_1\begin{bmatrix}1\\1\\0\end{bmatrix} + x_2\begin{bmatrix}2\\0\\4\end{bmatrix} = \begin{bmatrix}1\\2\\-2\end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 4 & -2 \end{array}\right]$

↑ no pivot in 3rd col.

So system is consistent!

A_2: yes, v_3 is in Span$\{v_1, v_2\}$ ✓

Linear Independence
Criterion

Suppose that one of the vectors $\{v_1, v_2, \ldots, v_p\}$ is a linear combination of the other ones (that is, it is in the span of the other ones):

$$v_3 = 2v_1 - \frac{1}{2}v_2 \iff 2v_1 - \frac{1}{2}v_2 - v_3 = 0 \text{ (dep. relation)}$$

Theorem
A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *dependent* if and only if one of the vectors is in the span of the other ones.

Equivalently:

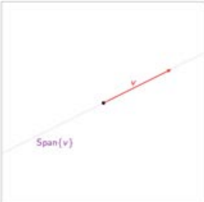
Theorem
A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *dependent* if and only if you can remove one of the vectors without shrinking the span.

Translation
A set of vectors is linearly independent if and only if, every time you add another vector to the set, the span gets bigger.

WARNING: does NOT say v_3 is in Span$\{v_1, v_2\}$ if and only if $\{v_1, v_2, v_3\}$ lin dep.

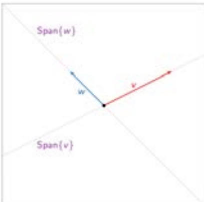
e.g. $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}$ are linearly dep but 3rd not in span of first two.

Linear Independence
Pictures in $\mathbf{R}^2$



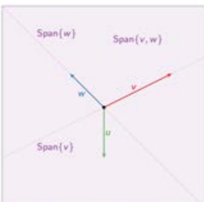
One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Linear Independence
Pictures in $\mathbf{R}^2$



One vector $\{v\}$:
Linearly independent if $v \neq 0$.
Two vectors $\{v, w\}$:
Linearly independent
- Neither is in the span of the other.
- Span got bigger.

Linear Independence
Pictures in $\mathbf{R}^2$



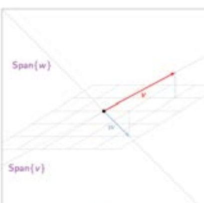
[interactive 2D: 2 vectors]
[interactive 2D: 3 vectors]

One vector $\{v\}$:
Linearly independent if $v \neq 0$.
Two vectors $\{v, w\}$:
Linearly independent
- Neither is in the span of the other.
- Span got bigger.

Three vectors $\{v, w, u\}$:
Linearly dependent:
- u is in Span$\{v, w\}$.
- Span didn't get bigger after adding u.
- Can remove u without shrinking the span.

Also v is in Span$\{u, w\}$ and w is in Span$\{u, v\}$.

Linear Independence
Pictures in $\mathbf{R}^3$

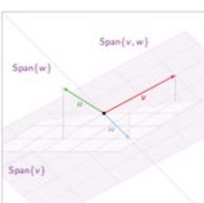


[interactive 3D: 2 vectors]
[interactive 3D: 3 vectors]

Two vectors $\{v, w\}$:
Linearly independent:
- Neither is in the span of the other.
- Span got bigger when we added w.

https://textbooks.math.gatech.edu/ila/demos/spans.html?captions=indep&v1=2,-1,1&v2=1,0,-1&v3=.5,-.5,1&labels=v,w,x&range=5

Linear Independence
Pictures in $\mathbf{R}^3$



[interactive 3D: 2 vectors]
[interactive 3D: 3 vectors]

Three vectors $\{v, w, u\}$:
Linearly independent: span got bigger when we added u.

$$\left[\begin{array}{cc|c} v & w & u \end{array}\right]$$

$x_1 v + x_2 w = u$ is an inconsistent system b/c pivot in last column.

know $\{v, w, u\}$ linearly independent, then u is not in span$\{v, w\}$

Linear Dependence and Free Variables

Theorem
Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$, and consider the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

Then you can delete the columns of A *without* pivots (the columns corresponding to free variables), without changing $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$. The pivot columns are linearly independent, so you can't delete any more columns.

This means that each time you add a pivot column, then the span increases.

Example: Which of the vectors in the set {v1,v2,v3,v4,v5} can be removed without changing the span of the vectors?

v1=[1;0;0], v2=[1;1;0], v3=[2;2;0], v4=[1;3;0], v5=[2;2;3]

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Q: give a new matrix B w/ some subset of cols of A as its cols, s.t.

(1) span of cols of B is same as span of cols of A.

(2) cols of B are lin ind.

Ans. (many answers possible to satisfy (1) & (2))

(a) $B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ ✓ keep pivot cols of A & erase other cols.

(b) $B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ ✓

(b') $B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ ✓

(c) $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ X

(d) $\begin{bmatrix} 2 & 1 & 2 \\ 2 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ ✓

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

Linear Independence
Two more facts

Fact 1: Say $v_1, v_2, \dots, v_n$ are in $\mathbf{R}^m$. If $n > m$ then $\{v_1, v_2, \dots, v_n\}$ is linearly dependent: the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

cannot have a pivot in each column (it is too wide).

This says you can't have 4 linearly independent vectors in $\mathbf{R}^3$, for instance.

A wide matrix can't have linearly independent columns.

Warning T F all tall matrices have linearly independent cols

Example: Is {v1,v2,v3} linearly independent or linearly dependent? If the set is linearly dependent, then find a dependence relation on v1,v2,v3.

v1=[1;1], v2=[0;1], v3=[2;3]

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}.$

$[A|0] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right]$ (columns are v_1, v_2, v_3) A is wide

Fact 2: If one of $v_1, v_2, \dots, v_n$ is zero, then $\{v_1, v_2, \dots, v_n\}$ is linearly dependent. For instance, if $v_1 = 0$, then

$$1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + \cdots + 0 \cdot v_n = 0$$

is a linear dependence relation.

A set containing the zero vector is linearly dependent.

Example: Is {v1,v2,v3} linearly independent or linearly dependent? If the set is linearly dependent, then find a dependence relation on v1,v2,v3.

v1=[1;1;0], v2=[0;0;0], v3=[0;1;1]

$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$ $v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$ $v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim$

Summary

- A set of vectors is **linearly independent** if removing one of the vectors shrinks the span; otherwise it's **linearly dependent**.
- There are several other criteria for linear (in)dependence which lead to pretty pictures.
- The columns of a matrix are linearly independent if and only if the RREF of the matrix has a pivot in every *column*.
- The pivot columns of a matrix A are linearly independent, and you can delete the non-pivot columns (the "free" columns) without changing the span of the columns.
- Wide matrices cannot have linearly independent columns.

Warning
These are not the official definitions!

Linear Independence

Definition
A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$. The set $\{v_1, v_2, \ldots, v_p\}$ is **linearly dependent** otherwise.

This is the official textbook definition.

Section 2.6

Subspaces

Motivation

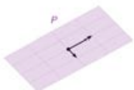
Today we will discuss **subspaces** of $\mathbf{R}^n$.

Examples

Example
A line L through the origin is a subspace: L contains zero and is easily seen to be closed under addition and scalar multiplication.

Example

A plane P through the origin is a subspace: P contains zero; the sum of two vectors in P is also in P; and any scalar multiple of a vector in P is also in P.



Example
All of $\mathbf{R}^n$: this contains 0, and is closed under addition and scalar multiplication.

Example
The subset $\{0\}$: this subspace contains only one vector.

Definition of Subspace

Definition
A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

A subspace is a span of some set of vectors in it.

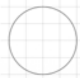
Subsets and Subspaces

They aren't the same thing

A **subset** of $\mathbf{R}^n$ is any collection of vectors in $\mathbf{R}^n$ whatsoever. For example, the unit circle

$$C = \{(x, y) \text{ in } \mathbf{R}^2 \mid x^2 + y^2 = 1\}$$

is a subset of $\mathbf{R}^2$, but it is not a subspace.



Subset: *yes*
Subspace: *no*

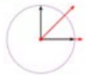
Non-Examples

Non-Example

A line L (or any other set) that doesn't contain the origin is not a subspace. Fails: 1.

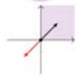
Non-Example

A circle C is not a subspace. Fails: 1,2,3. Think: a circle isn't a "linear space."



Non-Example

The first quadrant in $\mathbf{R}^2$ is not a subspace. Fails: 3 only.



Non-Example

A line union a plane in $\mathbf{R}^3$ is not a subspace. Fails: 2 only.

Subspaces are Spans, and Spans are Subspaces

Theorem

Any Span$\{v_1, v_2, \ldots, v_p\}$ is a subspace.

Definition

If $V = \text{Span}\{v_1, v_2, \ldots, v_p\}$, we say that V is the subspace **generated by** or **spanned by** the vectors $v_1, v_2, \ldots, v_p$. We call $\{v_1, v_2, \ldots, v_p\}$ a **spanning set** for V.

!!!

Every subspace is a span, and every span is a subspace.

Example: check from the definitions that Span{v1,v2} is a subspace where v1=[1;0;0] and v2=[0;1;0].

Poll

Poll

Which of the following are subspaces?

A. The empty set $\{\}$.

B. The solution set to a homogeneous system of linear equations.

C. The solution set to an inhomogeneous system of linear equations.

D. The set of all vectors in $\mathbf{R}^n$ with rational (fraction) coordinates.

For the ones which are not subspaces, which property(ies) do they not satisfy?

Poll

Poll

Which of the following are subspaces?

A. The empty set $\{\}$.

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D. The set of all vectors in $\mathbf{R}^n$ with rational (fraction) coordinates.

For the ones which are not subspaces, which property(ies) do they not satisfy?

Subspaces

Verification

Let $V = \left\{ \binom{a}{b} \text{ in } \mathbf{R}^2 \mid ab = 0 \right\}$. Let's check if V is a subspace or not.

Definition of Subspace

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Column Space and Null Space

An $m \times n$ matrix A naturally gives rise to *two* subspaces.

Definition

- The **column space** of A is the subspace of $\mathbf{R}^m$ spanned by the columns of A. It is written Col A.
- The **null space** of A is the set of all solutions of the homogeneous equation $Ax = 0$:

$$\text{Nul}\, A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$

This is a subspace of $\mathbf{R}^n$.

Column Space and Null Space

An $m \times n$ matrix A naturally gives rise to *two* subspaces.

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$$\text{Nul}\,A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$

This is a subspace of $\mathbf{R}^n$.

Example: Why is Col(A) always a subspace?

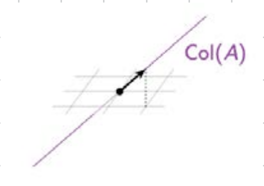
Example: Show that Nul(A) always satisfies all the conditions for being a subspace, using the definitions.

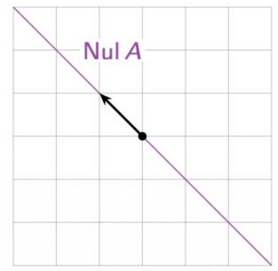
Column Space and Null Space

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Example: Find Col(A) and Nul(A) where A=[1 1 ; 1 1 ; 1 1]





Example: with A as above, describe Col(A) and Nul(A) geometrically.

The Null Space is a Span

The column space of a matrix A is defined to be a span (of the columns).

The null space is defined to be the solution set to $Ax = 0$. It is a subspace, so it is a span.

Question

How to find vectors that span the null space?

Example: Write the null space of the matrix A as a span.
A=[1 0 -1 2 ; 0 1 -2 -3]

Answer: Parametric vector form! We know that the solution set to $Ax = 0$ has a parametric form that looks like

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

if, say, x_3 and x_4 are the free variables. So

$$\operatorname{Nul} A = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Subspaces
Summary

- A **subspace** is the same as a span of some number of vectors, but we haven't computed the vectors yet.
- To any matrix is associated two subspaces, the **column space** and the **null space**:

$$\mathrm{Col}\,A = \text{the span of the columns of } A$$
$$\mathrm{Nul}\,A = \text{the solution set of } Ax = 0.$$

How do you check if a subset is a subspace?

- Is it a span? Can it be written as a span?
- Can it be written as the column space of a matrix?
- Can it be written as the null space of a matrix?
- Is it all of $\mathbf{R}^n$ or the zero subspace $\{0\}$?
- Can it be written as a type of subspace that we'll learn about later (eigenspaces, ...)?

If so, then it's automatically a subspace.

If all else fails:

- Can you verify directly that it satisfies the three defining properties?

Sections 2.7 and 2.9

Basis, Dimension, Rank and Basis Theorems

Subspaces
Reminder

Recall: a subspace of $\mathbf{R}^n$ is the same thing as a span, except we haven't computed a spanning set yet.

For example, Col A and Nul A for a matrix A.

There are lots of choices of spanning set for a given subspace.

Are some better than others?

Basis of a Subspace

What is the *smallest number* of vectors that are needed to span a subspace?

Definition
Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in V such that:
1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

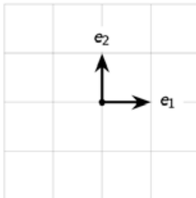
The number of vectors in a basis is the **dimension** of V, and is written dim V.

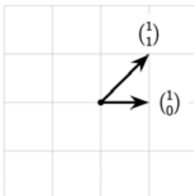
Note the big red border here

Important
A subspace has *many different* bases, but they all have the same number of vectors.

Bases of $\mathbf{R}^2$

Question
What is a basis for $\mathbf{R}^2$?





Bases of $\mathbf{R}^n$

The unit coordinate vectors

$$e_1 = \begin{pmatrix}1\\0\\\vdots\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\\vdots\\0\\0\end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix}0\\0\\\vdots\\1\\0\end{pmatrix}, \quad e_n = \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}$$

are a basis for $\mathbf{R}^n$. The identity matrix has columns $e_1, e_2, \ldots, e_n$.
1. They span: I_n has a pivot in every row.
2. They are linearly independent: I_n has a pivot in every column.

In general: $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^n$ if and only if the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

has a pivot in every row and every column.

Basis of a Subspace

Example

Example

Let

$$V=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\text{ in }\mathbf{R}^3 \mid x+3y+z=0\right\} \qquad \mathcal{B}=\left\{\begin{pmatrix}-3\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\-3\end{pmatrix}\right\}.$$

Verify that $\mathcal{B}$ is a basis for V. (So dim $V=2$: it is a *plane*.)

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=-3,1,0&v2=0,1,-3&range=5&captions=combo

Example: Find a basis for Nul(A) with A given below (the RREF of A is also given)

Basis for Nul A

Example

$$A=\begin{pmatrix}1&2&0&-1\\-2&-3&4&5\\2&4&0&-2\end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix}1&0&-8&-7\\0&1&4&3\\0&0&0&0\end{pmatrix}$$

Fact

The vectors in the parametric vector form of the general solution to $Ax=0$ always form a basis for Nul A.

Example: find a basis for Col(A) for the matrix A given below (the RREF of A is also given)

Basis for Col A

Example

$$A = \begin{pmatrix} \boxed{\begin{matrix}1\\-2\\2\end{matrix}} & \boxed{\begin{matrix}2\\-3\\4\end{matrix}} & \begin{matrix}0\\4\\0\end{matrix} & \begin{matrix}-1\\5\\-2\end{matrix} \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis $\longleftarrow$ pivot columns in rref

Fact

The *pivot columns* of A always form a basis for Col A.

Warning: I mean the pivot columns of the *original* matrix A, not the row-reduced form. (Row reduction changes the column space.)

The Basis Theorem

Basis Theorem

Let V be a subspace of dimension m. Then:

- Any m linearly independent vectors in V form a basis for V.
- Any m vectors that span V form a basis for V.

Example: any three linearly independent vectors form a basis for $\mathbf{R}^3$.

Upshot

If you *already* know that dim $V = m$, and you have m vectors $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ in V, then you only have to check *one of*

1. $\mathcal{B}$ is linearly independent, *or*
2. $\mathcal{B}$ spans V

in order for $\mathcal{B}$ to be a basis.

The Rank Theorem

Recall:

- The **dimension** of a subspace V is the number of vectors in a basis for V.
- A basis for the column space of a matrix A is given by the pivot columns.
- A basis for the null space of A is given by the vectors attached to the free variables in the parametric vector form.

Definition

The **rank** of a matrix A, written rank A, is the dimension of the column space Col A. The **nullity** of A, written nullity A, is the dimension of the solution set of $Ax = 0$.

Rank Theorem

If A is an $m \times n$ matrix, then

$$\operatorname{rank} A + \operatorname{nullity} A = n = \text{the number of columns of } A.$$

so that is:

(dimension of column space) + (dimension of solution set to Ax=0) = (number of variables)

The Rank Theorem

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

basis of Col A

free variables

You Try It!

Poll

Poll

True or False: If A is a 10×15 matrix and there is a basis of Col A consisting of 4 vectors, then there is a basis of Nul A consisting of 6 vectors.

Summary

- A **basis** of a subspace is a minimal set of spanning vectors.
- There are recipes for computing a basis for the column space and null space of a matrix.
- The **dimension** of a subspace is the number of vectors in any basis.
- The **basis theorem** says that if you already know that $\dim V = m$, and you have m vectors in V, then you only have to check if they span *or* they're linearly independent to know they're a basis.
- The **rank theorem** says the dimension of the column space of a matrix, plus the dimension of the null space, is the number of columns of the matrix.