

Sections 2.7 and 2.9

Basis, Dimension, Rank and Basis Theorems

Subspaces

Reminder

Recall: a subspace of $\mathbb{R}^n$ is the same thing as a span, except we haven't computed a spanning set yet.

For example, $\text{Col } A$ and $\text{Nul } A$ for a matrix A .

There are lots of choices of spanning set for a given subspace.

Are some better than others?

of A

how you choose to represent a subspace as a span of vectors is **NEVER** unique!

Solns to $Ax=0$

Basis of a Subspace

What is the smallest number of vectors that are needed to span a subspace?

Definition

Let V be a subspace of $\mathbb{R}^n$. A basis of V is a set of vectors $\{v_1, v_2, \dots, v_m\}$ in V such that:

1. $V = \text{Span}\{v_1, v_2, \dots, v_m\}$
2. $\{v_1, v_2, \dots, v_m\}$ is linearly independent

The number of vectors in a basis is the dimension of V , and is written $\dim V$.

Note the big red border here

Important

A subspace has many different bases, but they all have the same number of vectors.

Closed under vector add & scalar mult.

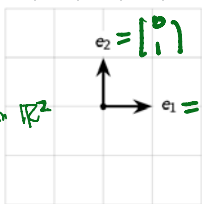
Bases of $\mathbb{R}^2$

Question

What is a basis for $\mathbb{R}^2$?

(1) $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$, for any $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ is always consistent. ✓

(2) $\{e_1, e_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ has a pivot in every col so $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent



Fact

$\mathbb{R}^n$ has dimension equal to n . So any basis for $\mathbb{R}^n$ has exactly n vectors in it.

e.g. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Bases of $\mathbb{R}^n$

$e_1, e_2, \dots, e_n$

The unit coordinate vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_{n-1} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

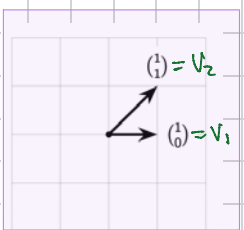
are a basis for $\mathbb{R}^n$. The identity matrix has columns $e_1, e_2, \dots, e_n$.

1. They span: I_n has a pivot in every row.
2. They are linearly independent: I_n has a pivot in every column.

In general: $\{v_1, v_2, \dots, v_n\}$ is a basis for $\mathbb{R}^n$ if and only if the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix} \begin{matrix} \text{size} \\ n \times n \end{matrix}$$

has a pivot in every row and every column.



Check (1) is $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ is consistent for every $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$

consistent b/c no pivot in any col.

(2) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ has a pivot in every col so $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is lin ind.

$$\sim \left(\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array} \right)$$

Punch line

cols are lin ind

every b is in span of $\{v_1, \dots, v_n\}$.

Basis of a Subspace

Example

Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x+3y+z=0 \right\} \quad B = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}$$

Verify that B is a basis for V . (So $\dim V = 2$: it is a plane.)

<https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=-3,1,0&v2=0,1,-3&range=5&captions=combo>

V is a plane in $\mathbb{R}^3$.

$$z + 3(-1) + 1 = 0 \quad \checkmark$$

Sanity check some vector in V are

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \checkmark$$

in V $\checkmark$

What else is in V ? is every vector in V a lin comb of $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$?

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V \quad \& \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -x-3y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \\ &= x \left(-\frac{1}{3} \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right) + y \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \\ &= -\frac{1}{3}x \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + \left(\frac{1}{3}x + y \right) \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \quad \checkmark \end{aligned}$$

Q: What is another basis

$$\text{for } V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x+3y+z=0 \right\}$$

The set V is just Solutions to the system $\begin{cases} x+3y+z=0 \end{cases}$

$$A = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}$$

$$x = -3y - z$$

$$y = y \text{ (free)}$$

$$z = z \text{ (free)}$$

$$x = y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So every soln to $Ax=0$ is a linear comb

$$\text{of } \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \left(\begin{array}{l} \text{from} \\ \text{param.} \\ \text{vector} \\ \text{form} \end{array} \right)$$

basis for V ?

yes they're lin and $\checkmark$

Example: Find a basis for $\text{Nul}(A)$ with A given below (the RREF of A is also given)

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Fact

The vectors in the parametric vector form of the general solution to $Ax=0$ always form a basis for $\text{Nul } A$.

Example: Find a basis for $\text{Nul}(A)$ with A given below
(the RREF of A is also given)

want a linearly independent spanning set for the solutions to $Ax=0$ homogeneous linear system.

Basis for $\text{Nul } A$

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

$$\text{rref} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

RREF of A

Fact

The vectors in the parametric vector form of the general solution to $Ax=0$ always form a basis for $\text{Nul } A$.

$$\begin{cases} x_1 = 8x_3 + 7x_4 \\ x_2 = -4x_3 - 3x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases} \rightarrow X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8x_3 + 7x_4 \\ -4x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} \\ = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

So the set of two vectors $\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\text{Nul } A$.

Q: $\text{Nul } A$ is a plane in $\mathbb{R}^4$.

Summary so far.

A basis for subspace V is a set of vectors $\{v_1, v_2, v_3\}$ s.t.

- ① $\{v_1, v_2, v_3\}$ span the subspace V
- ② $\{v_1, v_2, v_3\}$ are lin. ind.

Example: find a basis for $\text{Col}(A)$ for the matrix A given below (the RREF of A is also given)

Basis for Col A

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis $\longleftrightarrow$ pivot columns in rref

by def $\text{Col } A = \text{span} \left\{ \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -5 \\ 3 \end{pmatrix} \right\}$

Sanity Check: some vectors in $\text{Col } A$ are

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 1 \begin{pmatrix} -1 \\ -2 \end{pmatrix} + 0 \begin{pmatrix} -3 \\ 4 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 4 \end{pmatrix} + 0 \begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \in \text{Col } A$$

Not sure??

IDEA: extract pivot cols of original matrix A

Q1a: $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ is lin comb of $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ & $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

(2) $\left\{ \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix} \right\}$ lin ind?
 $\begin{pmatrix} -1 & -3 \\ 2 & 4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ consistent ✓

(1) $\left\{ \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix} \right\}$ spans $\text{Col } A$ ✓

IDEA: just extract the pivot cols of the ORIGINAL MATRIX.

These will be a basis for $\text{Col } A$.

✓

Fact

The pivot columns of A always form a basis for $\text{Col } A$.

Warning: I mean the pivot columns of the original matrix A , not the row-reduced form. (Row reduction changes the column space.)

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ also a basis for $\text{Col } A$?

(why 2?)

$\text{Col } A = \text{span} \left\{ \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -5 \\ 3 \end{pmatrix} \right\}$

↑ in general $A = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$ (size 3×4)

Then $\text{Col } A$ could be (a priori)

~~(1)~~ only $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

~~(2)~~ a line in $\mathbb{R}^3$

(3) a plane in $\mathbb{R}^3$

✓ (3) all of $\mathbb{R}^3$

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis $\longleftrightarrow$ pivot columns in rref

The Basis Theorem

Basis Theorem

Let V be a subspace of dimension m . Then:

- Any m linearly independent vectors in V form a basis for V .
- Any m vectors that span V form a basis for V .

Example: any three linearly independent vectors form a basis for $\mathbb{R}^3$.

Upshot

If you already know that $\dim V = m$, and you have m vectors $B = \{v_1, v_2, \dots, v_m\}$ in V , then you only have to check one of

- B is linearly independent, or
- B spans V

in order for B to be a basis.

The Rank Theorem

Recall:

- ▶ The **dimension** of a subspace V is the number of vectors in a basis for V .
- ▶ A basis for the column space of a matrix A is given by the pivot columns.
- ▶ A basis for the null space of A is given by the vectors attached to the free variables in the parametric vector form.

Definition

The **rank** of a matrix A , written $\text{rank } A$, is the dimension of the column space $\text{Col } A$. The **nullity** of A , written $\text{nullity } A$, is the dimension of the solution set of $Ax = 0$.

$$\text{rank}(A) = \# \text{ pivot} = \dim \text{Col } A.$$

$$\text{nullity}(A) = \# \text{ free vars} = \dim \text{Nul } A.$$

Rank Theorem

If A is an $m \times n$ matrix, then

$$\text{rank } A + \text{nullity } A = n = \text{the number of columns of } A.$$

so that is:

$$\# \text{ pivots} + \# \text{ free} = \# \text{ cols}$$

$$(\text{dimension of column space}) + (\text{dimension of solution set to } Ax=0) = (\text{number of variables})$$

Proof = each column of A is either pivot or free.

The Rank Theorem

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

basis of $\text{Col } A$

free variables

$$Ax = 0$$

$$x \in \mathbb{R}^4$$

$$\begin{pmatrix} -8 \\ 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 3 \\ 0 \\ 0 \end{pmatrix}$$

These were free cols
So $\dim \text{Nul } A = 2$

Q: are these in $\text{Nul } A$?



You Try It!

Poll

Poll

True or False: If A is a 10×15 matrix and there is a basis of $\text{Col } A$ consisting of 4 vectors, then there is a basis of $\text{Nul } A$ consisting of 6 vectors.

Summary

- ▶ A **basis** of a subspace is a minimal set of spanning vectors.
- ▶ There are recipes for computing a basis for the column space and null space of a matrix.
- ▶ The **dimension** of a subspace is the number of vectors in any basis.
- ▶ The **basis theorem** says that if you already know that $\dim V = m$, and you have m vectors in V , then you only have to check if they span or they're linearly independent to know they're a basis.
- ▶ The **rank theorem** says the dimension of the column space of a matrix, plus the dimension of the null space, is the number of columns of the matrix.

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis ← pivot columns in rref

extract THESE columns for a basis for Col A

NOT these columns!

Ex. Find a basis for $\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \right\} = V$ Subspace of $\mathbb{R}^3$

META: ① write

$$A = [v_1 \ v_2 \ v_3] = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$$V = \text{Col } A? \quad V = \text{Null } A?$$

② row reduce to REF (don't need RREF) to find pivots

③ extract pivot cols of ORIGINAL MATRIX (not the REF/rref!)

$$\text{Col } A = \text{"span of cols of } A" = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \right\} = V$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

basis for Col A is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ So

$$V = \text{Col } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ which is a line in } \mathbb{R}^3$$

$$\text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \neq \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

also a line in $\mathbb{R}^3$

Chapter 3

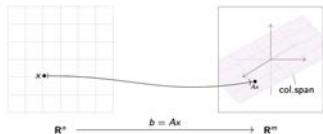
Linear Transformations and Matrix Algebra

Section 3.1

Matrix Transformations

Motivation

Let A be a matrix, and consider the matrix equation $b = Ax$. If we vary x , we can think of this as a function of x .



$$\begin{aligned} f(x) &= y = x^2 \\ g(x) &= y = 3x \end{aligned}$$

rules which take an input (number)

and DO SOMETHING to get an output (number)

$$\begin{array}{c|c} x & y \\ \hline -1 & -3 \\ 0 & 0 \\ 1 & 3 \\ 2 & 6 \end{array}$$

Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, plot some inputs x and outputs Ax

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \textcircled{1} \text{ For some } x \text{ in } \mathbb{R}^2, \text{ compute } Ax.$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ input vector gives $\begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$ output vector

Example: with A as below, plot some inputs and outputs

Matrices as Functions
Projection

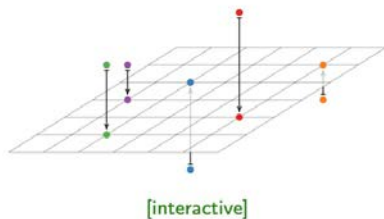
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

[https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?](https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?mat=1,0,0,0,1,0,0,0,0&range2=5&closed=true)

mat=1,0,0,0,1,0,0,0,0&range2=5&closed=true

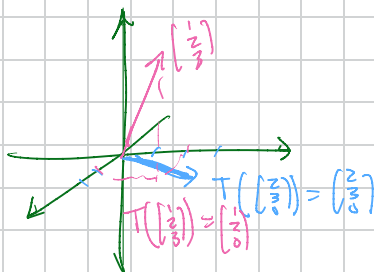
Now find some "random" inputs x in $\mathbb{R}^3$ & compute the output Ax .

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 + 0 + 0 \\ 0 + 3 + 0 \\ 0 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

projection onto xy-plane



"T" function name

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

Example: for the matrix A given below, plot some input and output vectors for $T(x)=Ax$, and give the function T an appropriate NAME.

<https://textbooks.math.gatech.edu/ila/demos/twobytwo.html?mat=-1,0,0,1&closed=true>

Matrices as Functions
Reflection

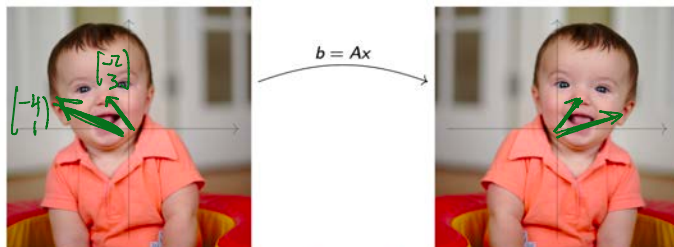
$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} (-1)(-4) + 0(1) \\ 0(-4) + 1(1) \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ b \end{pmatrix}$$

Example: plot some input and output vectors for $T(x)=Ax$ and give the function T a NAME.



[interactive]

x-value flipper / horizontal mirrorer
 Reflection over y-axis
 (horizontal reflection)

<https://textbooks.math.gatech.edu/ila/demos/twobytwo.html?mat=1.5,0,0,1.5&closed=true&pic=theo3.jpg>

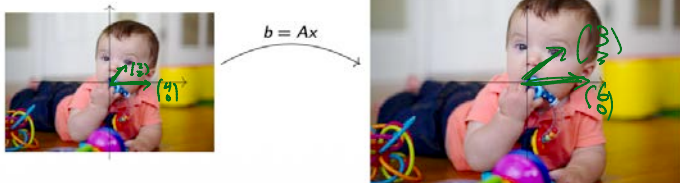
Matrices as Functions
Dilation

$$A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1.5a \\ 1.5b \end{pmatrix}$$



[interactive]

The vector elongator.

1.5x zoom

Scaling / expansion
 The bigger-er

Example: ditto

Matrices as Functions

Identity

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the identity matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

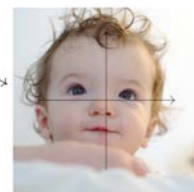
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

<https://textbooks.math.gatech.edu/ila/demos/twobytwo.html?mat=1,0,0,1&closed=true&pic=theo11.jpg>



$$b = Ax$$



[interactive]

The do nothing transformation.
nothing moves.

the identity transformation

Matrices as Functions

Rotation

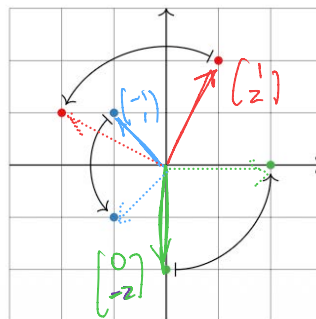
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0-2 \\ 1+0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-1 \\ -1+0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0-(-2) \\ 0+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

rotation by 90° counter-clockwise (CCW)



<https://textbooks.math.gatech.edu/ila/demos/twobytwo.html?mat=0,-1,1,0&closed=true&pic=theo8.jpg>



$$b = Ax$$



[interactive]

Other Geometric Transformations

In §3.1 there are other examples of geometric transformations of $\mathbb{R}^2$ given by matrices. Please look them over.

Transformations

Motivation

We have been drawing pictures of what it looks like to multiply a matrix by a vector, as a function of the vector.

Now let's go the other direction. Suppose we have a function, and we want to know, does it come from a matrix?

Example

For a vector x in $\mathbb{R}^2$, let $T(x)$ be the counterclockwise rotation of x by an angle θ . Is $T(x) = Ax$ for some matrix A ?

If $\theta = 90^\circ$, then we know $T(x) = Ax$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

But for general θ , it's not clear.

Our next goal is to answer this kind of question.

The matrix A is doing something
to input vectors to get output vectors

Transformations

Vocabulary

Definition

A transformation (or function or map) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule T that assigns to each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$.

$\mathbb{R}^n$ is called the **domain** of T (the inputs).

$\mathbb{R}^m$ is called the **codomain** of T (where the outputs live).

For x in $\mathbb{R}^n$, the vector $T(x)$ in $\mathbb{R}^m$ is the **image** of x under T .

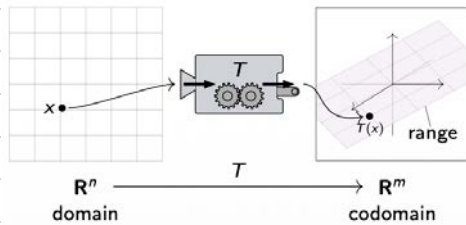
Notation: $x \mapsto T(x)$.

The set of all images $\{T(x) \mid x \text{ in } \mathbb{R}^n\}$ is the **range** of T .

Notation

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ means T is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

particular output vector
set of all output vectors.



Where inputs live

Where outputs live

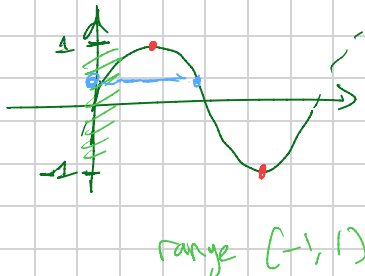
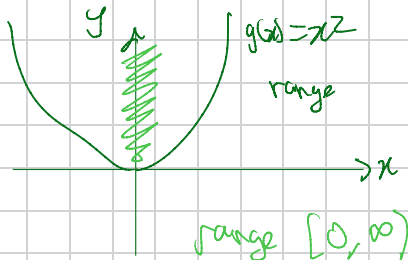
Examples from calculus: $f(x) = \sin(x)$ and $g(x) = x^2$

$g: \mathbb{R} \rightarrow \mathbb{R}$

Codomain is $\mathbb{R}$

$f: \mathbb{R} \rightarrow \mathbb{R}$

$f(x) = \sin x$



Functions from Engineering

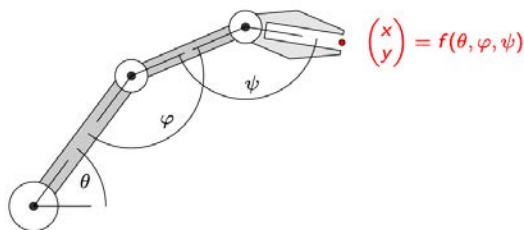
Suppose you are building a robot arm with three joints that can move its hand around a plane, as in the following picture.

Matrix Transformations

Definition

Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T: \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ defined by } T(x) = Ax.$$



Example: find the domain, codomain, and range of the matrix transformation $T(x)$ below

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 5 \\ -3 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Handwritten notes: A has 3 rows, A has 3 cols. $\uparrow$ input in $\mathbb{R}^3$, $\uparrow$ output in $\mathbb{R}^3$.

- ① domain is $\mathbb{R}^3$
- ② codomain is $\mathbb{R}^3$
- ③ range is $\mathbb{R}^3$

where inputs live

- ▶ The domain of T is $\mathbb{R}^n$, which is the number of columns of A .
- ▶ The codomain of T is $\mathbb{R}^m$, which is the number of rows of A .
- ▶ The range of T is the set of all images of T :

$$T(x) = Ax = \begin{pmatrix} | & | & \dots & | \\ v_1 & v_2 & \dots & v_n \\ | & | & \dots & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \dots + x_n v_n.$$

where outputs live.

range?? what vectors $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ are outputs that are obtainable?

$$x \begin{bmatrix} 1 \\ 4 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \end{bmatrix} + z \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

when is this vector in the span of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

Example: find the domain, codomain, and range of the matrix transformation $T(x)=Ax$ with the matrix A given below

$$A = \begin{pmatrix} -1 & 0 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$$

$$T(x) = Ax$$

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

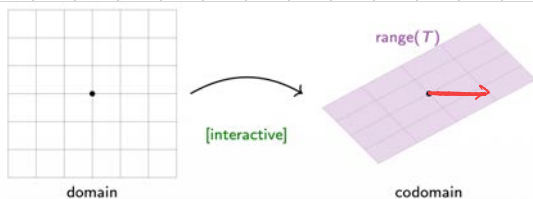
$\uparrow$ domain (inputs) $\uparrow$ codomain (outputs)

when no pivot in aug col

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \end{array} \right]$$

Handwritten notes: no pivot for $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. any $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

<https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?mat=-1,0;2,1;1,-1&closed=true&show=false>



$$\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\uparrow ?$ $\uparrow ?$

domain $\mathbb{R}^2$

codomain $\mathbb{R}^3$

range? Span $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ a plane in $\mathbb{R}^3$.

You Try It!

Poll

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$. (T is called a shear.)

Find: the domain, the codomain, and the range of T .

Also plot a few input and output vectors, and give the transformation a NAME

domain $\mathbb{R}^2$

codomain $\mathbb{R}^2$

range $\text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} = \mathbb{R}^2$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ z \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \end{pmatrix}$$

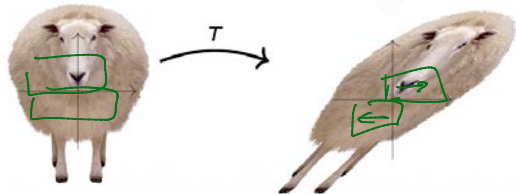
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ b \end{pmatrix}$$

Name: xy adder and second number divider.

(Shear.)

Summary

- We can think of $b = Ax$ as a **transformation** with input x and output b .
- There are vocabulary words associated to transformations: **domain**, **codomain**, **range**.
- A transformation that comes from a matrix is called a **matrix transformation**.
- In this case, the vocabulary words mean something concrete in terms of matrices.
- We like transformations that come from matrices, because questions about those transformations turn into questions about matrices.



Section 3.2

One-to-one and Onto Transformations

Matrix Transformations

Reminder

Recall: Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ defined by } T(x) = Ax.$$

- The domain of T is $\mathbb{R}^n$, which is the number of columns of A .
- The codomain of T is $\mathbb{R}^m$, which is the number of rows of A .
- The range of T is the set of all images of T :

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

This is the column space of A . It is a span of vectors in the codomain.

4

Matrix Transformations

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

(1) find $T(u)$ where $u = [3; 4]$

(2) let $b = [7; 5; 7]$, find v in $\mathbb{R}^2$ such that $T(v) = b$.

(3) is there any other vector w (other than v) that also gets sent to b ?

1,

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

Same A as previous example.

(4) is there any vector c in $\mathbb{R}^3$ which is not in the range of T ? In other words, can you find a vector c for which NO vector v in $\mathbb{R}^2$ maps to c ?

Translation: Find c such that $Ax = c$ is inconsistent.

Matrix Transformations

Note: All of these questions are questions about the transformation T ; it still makes sense to ask them in the absence of the matrix A .

The fact that T comes from a matrix means that these questions translate into questions about a matrix, which we know how to do.

Non-example: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$

We will restrict our focus to matrix transformations in this class, but it is good to know that you can still ask similar questions about arbitrary transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Questions About Transformations

Today we will focus on two important questions one can ask about a transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$:

- Do there exist distinct vectors x, y in $\mathbb{R}^n$ such that $T(x) = T(y)$?
- For every vector y in $\mathbb{R}^m$, does there exist a vector x in $\mathbb{R}^n$ such that $T(x) = y$?

These are subtle because of the multiple quantifiers involved ("for every", "there exists").

One-to-one Transformations

Definition

A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one-to-one** (or **into**, or **injective**) if different vectors in $\mathbb{R}^n$ map to different vectors in $\mathbb{R}^m$. In other words, for every b in $\mathbb{R}^m$, the equation $T(x) = b$ has at most one solution x . Or, different inputs have different outputs. Note that not one-to-one means at least two different vectors in $\mathbb{R}^n$ have the same image.

Matrix Transformations

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

Same example as before: Is the transformation $T(x) = Ax$ one-to-one?

Characterization of One-to-One Matrix Transformations

Theorem

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a matrix transformation with matrix A . Then the following are equivalent:

- T is one-to-one.
- For each b in $\mathbb{R}^m$, the equation $T(x) = b$ has at most one solution.
- For each b in $\mathbb{R}^m$, the equation $Ax = b$ has a unique solution or is inconsistent.
- $Ax = 0$ has a unique solution.
- The columns of A are linearly independent.
- A has a pivot in every column.

Question

If $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one, what can we say about the relative sizes of n and m ?

What can you say about A ? $n \leq m$ (#cols $\leq$ #rows)

What sizes can A be? can be TALL

What sizes can A NOT be? can NOT be WIDE

New example:

One-to-One Transformations

Example

Define

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad T(x) = Ax,$$

so $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$. Is T one-to-one?

- (1) domain of T is
- (2) codomain of T is
- (3) range of T is a line in $\mathbb{R}^3$
- (4) is T one-to-one?

Non-example:

One-to-One Transformations

Non-Example

Define

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad T(x) = Ax,$$

- (1) Is the transformation $T(x)=Ax$ one-to-one?
- (2) If not, find two **different** vectors v and w that map to the **same** output b
- (3) Also, is **every** vector b in $\mathbb{R}^2$ the output for some input vector v ?

One-to-one Transformations

Definition

A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one-to-one** (or **into**, or **injective**) if different vectors in $\mathbb{R}^n$ map to different vectors in $\mathbb{R}^m$. In other words, for every b in $\mathbb{R}^m$, the equation $T(x) = b$ has at most one solution x . Or, different inputs have different outputs. Note that **not one-to-one** means at least two different vectors in $\mathbb{R}^n$ have the same image.

Onto Transformations

Definition

A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **onto** (or **surjective**) if the range of T is equal to $\mathbb{R}^m$ (its codomain). In other words, for every b in $\mathbb{R}^m$, the equation $T(x) = b$ has at least one solution. Or, every possible output has an input. Note that **not onto** means there is some b in $\mathbb{R}^m$ which is not the image of any x in $\mathbb{R}^n$.

Same example as before:

One-to-One Transformations

Non-Example

Define

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad T(x) = Ax.$$

(1) is $T(x) = Ax$ onto?

Characterization of Onto Matrix Transformations

Theorem

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a matrix transformation with matrix A . Then the following are equivalent:

- ▶ T is onto
- ▶ $T(x) = b$ has a solution for every b in $\mathbb{R}^m$
- ▶ $Ax = b$ is consistent for every b in $\mathbb{R}^m$
- ▶ The columns of A span $\mathbb{R}^m$
- ▶ A has a pivot in every row

Same example as before:

Matrix Transformations

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

(1) is $T(x) = Ax$ onto?

Question

If $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is onto, what can we say about the relative sizes of n and m ?

* IF A is $m \times n$ then $m \leq n$ (# rows $\leq$ # cols)

* A can be WIDE for T to be onto

* A can NOT be tall for T to be onto.

Last example:

One-to-One and Onto Transformations

Non-Example

Define

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad T(x) = Ax,$$

so $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$. Is T one-to-one? Is it onto?

(1) find the domain and codomain of T

(2) is T one-to-one?

(3) is T onto?

Summary

- ▶ A transformation T is **one-to-one** if $T(x) = b$ has at most one solution, for every b in $\mathbb{R}^n$.
- ▶ A transformation T is **onto** if $T(x) = b$ has at least one solution, for every b in $\mathbb{R}^n$.
- ▶ A matrix transformation with matrix A is one-to-one if and only if the columns of A are linearly independent, if and only if A has a pivot in every column.
- ▶ A matrix transformation with matrix A is onto if and only if the columns of A span $\mathbb{R}^n$, if and only if A has a pivot in every row.
- ▶ Two of the most basic questions one can ask about a transformation is whether it is one-to-one or onto.