

Section 3.3

Linear Transformations

Linear Transformations

Motivation

In the last two lectures we have been asking questions about transformations, and answering them in the case of matrix transformations.

However, sometimes it is not clear if a transformation is a matrix transformation or not.

Example

For a vector x in $\mathbb{R}^2$, let $T(x)$ be the counterclockwise rotation of x by an angle θ . Is $T(x) = Ax$ for some matrix A ?



Linear Transformations

Definition

A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **linear** if it satisfies the equations for all vectors u, v in $\mathbb{R}^n$ and all scalars c .

(1) $T(u+v) = T(u) + T(v)$ "T respects addition"

(2) $T(cv) = cT(v)$ "T respects scalar multiplication"

T is linear if (1) & (2) hold.

Some immediate consequences:

Check: if T is linear, then

$$T(0) = 0 \quad T(cu + dv) = cT(u) + dT(v)$$

for all vectors u, v and scalars c, d . More generally,

$$T(c_1 v_1 + c_2 v_2 + \dots + c_n v_n) = c_1 T(v_1) + c_2 T(v_2) + \dots + c_n T(v_n).$$

In engineering this is called **superposition**.

if you apply a linear transformation to a linear combination of vectors

you get the linear combination of the images w/ same scalar.

Linear Transformations

Dilation

Define $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(x) = 1.5x$. Is T linear? Check:

Example

need to check (1) $T(u+v) = T(u) + T(v)$
(2) $T(cu) = cT(u)$

e.g. $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 1.5\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 3 \end{bmatrix}$

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = 1.5\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1.5a \\ 1.5b \end{bmatrix}$$

$$T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) = 1.5\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1.5c \\ 1.5d \end{bmatrix}$$

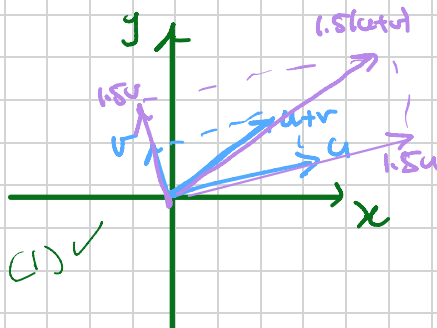
$$T(u+v) = T\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) = T\left(\begin{bmatrix} a+c \\ b+d \end{bmatrix}\right) = 1.5\begin{bmatrix} a+c \\ b+d \end{bmatrix} = \begin{bmatrix} 1.5a + 1.5c \\ 1.5b + 1.5d \end{bmatrix}$$

$$\begin{aligned} (2) \quad T\left(k\begin{bmatrix} a \\ b \end{bmatrix}\right) &= 1.5k\begin{bmatrix} a \\ b \end{bmatrix} \quad (2) \checkmark \\ &= k \cdot 1.5\begin{bmatrix} a \\ b \end{bmatrix} = kT\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \\ &= T(u) + T(v) \end{aligned}$$

Note: T is a matrix transformation!

$$T(x) = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} x,$$

as we checked before.



Linear Transformations

Rotation

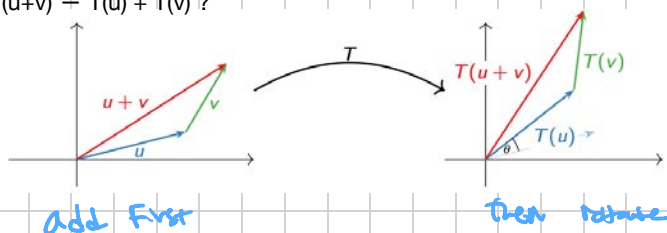
Define $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$T(x)$ = the vector x rotated counterclockwise by an angle of θ .

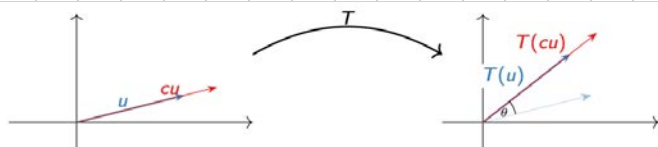
Is T linear? Check:

Proof by picture:

(1) $T(u+v) \stackrel{?}{=} T(u) + T(v)$?



(2) $T(cu) \stackrel{?}{=} c T(u)$?



$$T(cu) = c T(u)$$

↑
scale first, then rotate

rotate first, then scale

Linear Transformations

Non-example

Is every transformation a linear transformation?

No! For instance, $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$ is not linear.

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

Hint: check what happens to $T(v)$ if $v=0$ the zero vector.

easy way to check if T is linear!

$$T \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin 0 \\ 0 \cdot 0 \\ \cos 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so T can't be linear! why? $T(0) = 0$

would have to be true if T was linear

You Try It!

Poll

Which of the following transformations are linear?

~~A.~~ $T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} |x_1| \\ x_2 \end{pmatrix}$ $\checkmark$ B. $T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_1 - 2x_2 \end{pmatrix}$

~~C.~~ $T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ x_2 \end{pmatrix}$ ~~D.~~ $T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + 1 \\ x_1 - 2x_2 \end{pmatrix}$

(D) $T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 1 \\ 1 - 2 \cdot 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\therefore$ (D) T is not linear.

(B) IS linear but lin hold on the proof

(A) $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $v = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$T(u+v) \neq T(u) + T(v) = T \begin{pmatrix} 1 \\ 1 \end{pmatrix} + T \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$\hookrightarrow T \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) = T \left(\begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ this T is Not linear.

$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $c = 2$

(G) $T(cu) \neq cT(u) = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$\hookrightarrow T \left(2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = T \left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

T is Not linear.

The Matrix of a Linear Transformation

We will see that a linear transformation T is a matrix transformation:
 $T(x) = Ax$.

But what matrix does T come from? What is A ?

Here's how to compute it.

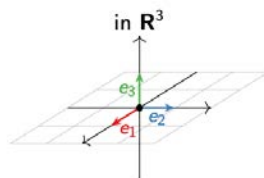
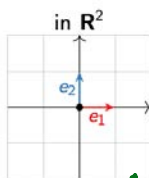
The Matrix of a Linear Transformation

We will see that a linear transformation T is a matrix transformation:
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But what matrix does T come from? What is A ?

Here's how to compute it.

How to find A ?



First: recall the definitions of $e_1, e_2, \dots, e_n$

Unit Coordinate Vectors

Definition

The unit coordinate vectors in $\mathbb{R}^n$ are

This is what $e_1, e_2, \dots, e_n$ mean for the rest of the class.

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_{n-1} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

aka standard basis vectors in $\mathbb{R}^n$.

in $\mathbb{R}^2$
 $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
 in $\mathbb{R}^3$
 $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ compute Ae_1 , Ae_2 , and Ae_3 .

$$Ae_1 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$$

Ae_1 equals the first column of A .

$$Ae_2 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$$

Ae_2 is the second column of A .

$$Ae_3 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

Ae_3 is the third column of A .

Note: if A is an $m \times n$ matrix with columns $v_1, v_2, \dots, v_n$, then $Ae_i = v_i$ for $i = 1, 2, \dots, n$: multiplying a matrix by e_i gives you the i th column.

transformations are linear?

$$B. T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_1 - 2x_2 \end{pmatrix}$$

What's A ?

$$T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2+0 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+1 \\ 0-2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

First column is $T(e_1) = Ae_1$
 Second column is $T(e_2) = Ae_2$

Linear Transformations are Matrix Transformations

Theorem
 Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Let

$$A = \begin{pmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{pmatrix}$$

This is an $m \times n$ matrix, and T is the matrix transform. The matrix A is called the standard matrix for T .

Take-Away

Linear transformations are the same as matrix transformations.

$$T(e_1) = Ae_1 = \text{first col of } A$$

$$T(e_2) = Ae_2 = \text{second col of } A$$

$$T(e_3) = Ae_3 = \text{third col of } A$$

most important info for the day.

Linear Transformations are Matrix Transformations

Example
Before, we defined a dilation transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(x) = 1.5x$.
What is its standard matrix?

How to get A ? Apply T to $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

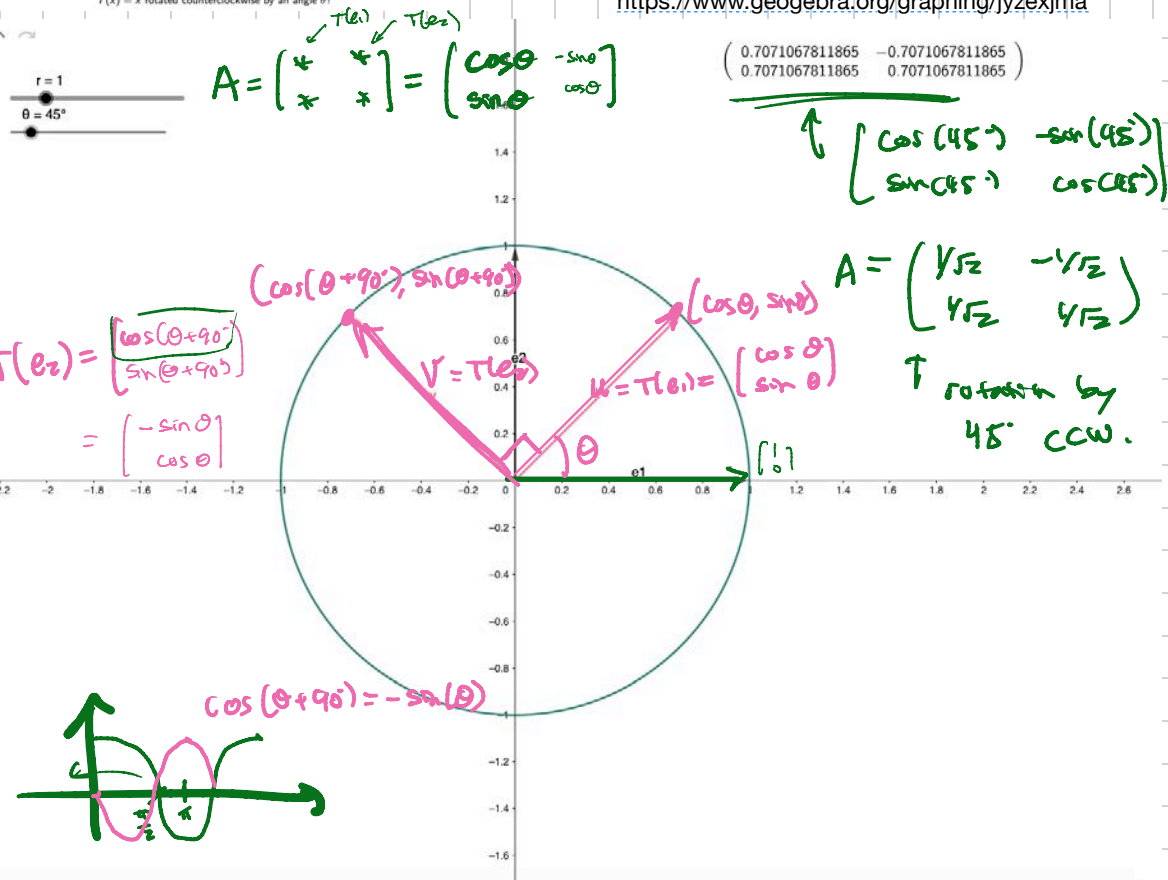
$T(e_1) \rightarrow T(e_2)$

$$A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}$$

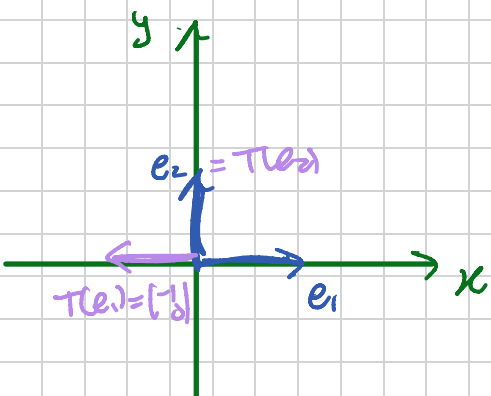
Linear Transformations are Matrix Transformations

Question
What is the matrix for the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by
 $T(x) = x$ rotated counterclockwise by an angle θ ?

<https://www.geogebra.org/graphing/jyzexjma>



Ex. Find A for reflect across y -axis
in $\mathbb{R}^2$



$$T(e_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = [T(e_1) \ T(e_2)] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x$$

do horizontal reflection
to x

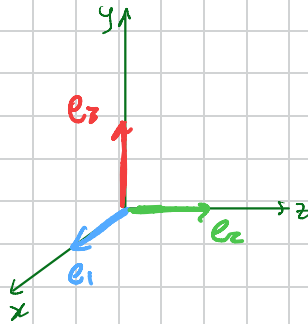
Linear Transformations are Matrix Transformations

Example, continued

Question

What is the matrix for the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that reflects through the xy -plane and then projects onto the yz -plane?

- (1) Find the standard matrix A of T
- (2) is T one-to-one?
- (3) is T onto?



Linear Transformations are Matrix Transformations

Example

Question

Define a linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ -y - 5z \end{pmatrix}.$$

What is the standard matrix A for T ?

- (1) Find the standard matrix A of T
- (2) is T one-to-one?
- (3) is T onto?

Summary

- Linear transformations are the transformations that come from matrices.
- The **unit coordinate vectors** $e_1, e_2, \dots$ are the unit vectors in the positive direction along the coordinate axes.
- You compute the columns of the matrix for a linear transformation by plugging in the unit coordinate vectors.
- This is useful when the transformation is specified geometrically, in terms of a formula, or any other way that isn't as a matrix transformation.

$$\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \quad n < m \quad (\text{tall})$$

$$\begin{bmatrix} * & * \\ * & * \end{bmatrix} \quad n = m \quad (\text{square})$$

$$\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix} \quad n > m \quad (\text{wide})$$

Section 3.4

Matrix Multiplication

Motivation

Recall: we can turn any system of linear equations into a matrix equation

$$Ax = b.$$

This notation is suggestive. Can we solve the equation by "dividing by A "?

$$x \stackrel{??}{=} \frac{b}{A}$$

Answer: Sometimes, but you have to know what you're doing.

Today we'll study *matrix algebra*: adding and multiplying matrices.

These are not so hard to do. The important thing to understand today is the relationship between *matrix multiplication* and *composition of transformations*.

More Notation for Matrices

Let A be an $m \times n$ matrix.

We write a_{ij} for the entry in the i th row and the j th column. It is called the i th entry of the matrix.

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

j th column

The entries $a_{11}, a_{22}, a_{33}, \dots$ are the **diagonal entries**; they form the **main diagonal** of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

A **diagonal matrix** is a square matrix whose only nonzero entries are on the main diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for all v in $\mathbb{R}^n$.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

"acting."

More Notation for Matrices

Continued

The **zero matrix** (of size $m \times n$) is the $m \times n$ matrix 0 with all zero entries.

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A . In other words, the j th entry of A^T is a_{ji} .

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \rightsquigarrow A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

$\nearrow$ flip

Matrix Multiplication

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix with columns $v_1, v_2, \dots, v_p$:

must be equal

$$B = \begin{pmatrix} | & | & v_2 & \cdots & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & \end{pmatrix}$$

The product AB is the $m \times p$ matrix with columns $Av_1, Av_2, \dots, Av_p$:

The equality is a definition

$$AB \stackrel{\text{def}}{=} \begin{pmatrix} | & | & Av_2 & \cdots & | \\ Av_1 & Av_2 & \cdots & Av_p \\ | & | & \end{pmatrix}$$

In order for $Av_1, Av_2, \dots, Av_p$ to make sense, the number of columns of A has to be the same as the number of rows of B . Note the sizes of the product!

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} \\ = \begin{pmatrix} 14 & -10 \\ 32 & -28 \end{pmatrix}$$

The ij entry of $C = AB$ is the i th row of A times the j th column of B :

$$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$\begin{pmatrix} a_{i1} & \cdots & a_{i1} & \cdots & a_{in} \\ a_{11} & \cdots & a_{1n} & \cdots & a_{nn} \\ \vdots & & \vdots & & \vdots \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & & \vdots & & \vdots \end{pmatrix} = \begin{pmatrix} c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \end{pmatrix}$$

j th column i j entry

The Row-Column Rule for Matrix Multiplication

Example

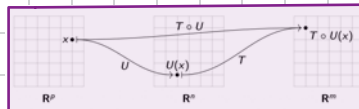
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 & \square \\ \square & \square \end{pmatrix} = \begin{pmatrix} 14 & \square \\ \square & \square \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \square & \square \\ 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 & \square \end{pmatrix} = \begin{pmatrix} \square & \square \\ 32 & \square \end{pmatrix}$$

Composition of Transformations

Definition

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $U: \mathbb{R}^2 \rightarrow \mathbb{R}^n$ be transformations. The composition is the transformation

$$T \circ U: \mathbb{R}^2 \rightarrow \mathbb{R}^m \quad \text{defined by} \quad T \circ U(x) = T(U(x)).$$



The matrix of the composition is the product of the matrices!

Example: Find the standard matrix for the transformation $T \circ U$ where

U is the transformation which rotates vectors in $\mathbb{R}^2$ by 90° clockwise, and

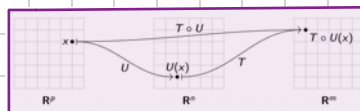
T is the transformation which projects vectors in $\mathbb{R}^2$ to the x -axis.

Composition of Transformations

Definition

Let $T: \mathbb{R}^k \rightarrow \mathbb{R}^m$ and $U: \mathbb{R}^n \rightarrow \mathbb{R}^k$ be transformations. The **composition** is the transformation

$$T \circ U: \mathbb{R}^n \rightarrow \mathbb{R}^m \quad \text{defined by} \quad T \circ U(x) = T(U(x)).$$



The matrix of the composition is the product of the matrices!

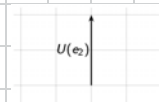
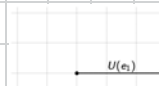
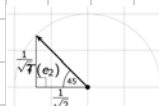
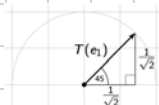
Example: Find the standard matrix for the transformation $T \circ U$ where U is the transformation which rotates vectors in $\mathbb{R}^2$ by 90° *clockwise*, and T is the transformation which projects vectors in $\mathbb{R}^2$ to the x-axis.

Composition of Linear Transformations

Another Example

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be rotation by 45° , and let $U: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ scale the x -coordinate by 1.5. Let's compute their standard matrices A and B :

Also: find the standard matrix for $G = T \circ U$



You Try It!

Poll

Poll

Do there exist nonzero matrices A and B with $AB = 0$?

Hint: think geometrically!

You Try It!

Composition of Linear Transformations

Example

Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $U: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the matrix transformations

$$T(x) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} x \quad U(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} x.$$

A B

Example: What is the standard matrix for the transformation which first applies the transformation T to a vector x and then applies the transformation U to the result?

Addition and Scalar Multiplication for Matrices

You add two matrices component by component, like with vectors.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{pmatrix}$$

Note you can only add two matrices of the same size.

You multiply a matrix by a scalar by multiplying each component, like with vectors.

$$c \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{pmatrix}.$$

These satisfy the expected rules, like with vectors:

$$\begin{aligned} A + B &= B + A & (A + B) + C &= A + (B + C) \\ c(A + B) &= cA + cB & (c + d)A &= cA + dA \\ (cd)A &= c(dA) & A + 0 &= A \end{aligned}$$

Properties of Matrix Multiplication

Mostly matrix multiplication works like you'd expect. Suppose A has size $m \times n$, and that the other matrices below have the right size to make multiplication work.

$$\begin{aligned} A(BC) &= (AB)C & A(B + C) &= (AB + AC) \\ (B + C)A &= BA + CA & c(AB) &= (cA)B \\ c(AB) &= A(cB) & I_m A &= A \\ AI_n &= A \end{aligned}$$

Warnings!

- AB is usually not equal to BA .

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =$$

Warnings!

- $AB = AC$ does not imply $B = C$, even if $A \neq 0$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix} =$$

Warnings!

- $AB = 0$ does not imply $A = 0$ or $B = 0$.

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} =$$

Powers of a Matrix

Definition

Let n be a positive whole number and let A be a square matrix. The n th power of A is the product

$$A^n = \underbrace{A \cdot A \cdot \dots \cdot A}_{n \text{ times}}$$

Example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A^2 =$$

$$A^3 =$$

Summary

- The product of an $m \times n$ matrix and an $n \times p$ matrix is an $m \times p$ matrix. I showed you two ways of computing the product.
- Composition of linear transformations corresponds to multiplication of matrices.
- You have to be careful when multiplying matrices together, because things like commutativity and cancellation fail.
- You can take powers of square matrices.

Section 3.5 and 3.6

Matrix Inverses and the Invertible Matrix Theorem

The Definition of Inverse

Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that

$$AB = I_n \text{ and } BA = I_n$$

identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

In this case, B is the **inverse** of A , and is written A^{-1} .

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

I claim $B = A^{-1}$. Check:

Compare to numbers.

3 is invertible (using multiplication)

$$\text{etc } 3 * \frac{1}{3} = 1.$$

You Try It!

Poll

Do there exist two matrices A and B such that AB is the identity, but BA is not? If so, find an example. (Both products have to make sense.)

Hint: don't assume A and B have the same size.

However

If A and B are square matrices, then

$$AB = I_n \text{ if and only if } BA = I_n.$$

So in this case you only have to check one.

Key observation: Having an inverse of a coefficient matrix of a system $Ax=b$ makes solving the system very easy!

Solving Linear Systems via Inverses

Example

Solve the system

$$\begin{array}{rcl} 2x_1 + 3x_2 + 2x_3 & = & 1 \\ x_1 & + & 3x_3 = 1 \\ 2x_1 + 2x_2 + 3x_3 & = & 1 \end{array} \quad \text{using} \quad \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}.$$

Solve
 $Ax=b$

$$A = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$2x_1 + 3x_2 + 2x_3 = 1$$

$$x_1 + 3x_3 = 1$$

$$2x_1 + 2x_2 + 3x_3 = 1$$

The advantage of using inverses is it doesn't matter what's on the right-hand side of the = :

$$\begin{cases} 2x_1 + 3x_2 + 2x_3 = b_1 \\ x_1 + 3x_3 = b_2 \\ 2x_1 + 2x_2 + 3x_3 = b_3 \end{cases} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -6b_1 - 5b_2 + 9b_3 \\ 3b_1 + 2b_2 - 4b_3 \\ 2b_1 + 2b_2 - 3b_3 \end{pmatrix}$$

Computing A^{-1}

The 2×2 case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **determinant** of A is the number

$$\det(A) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Facts:

1. If $\det(A) \neq 0$, then A is invertible and $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
2. If $\det(A) = 0$, then A is not invertible.

Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} =$$

Invertible Linear Transformations

Example

Let T = counterclockwise rotation in the plane by 45° . Its matrix is

$$A = \begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then T^{-1} = counterclockwise rotation by -45° . Its matrix is what matrix?

Let T = counterclockwise rotation in the plane by 45° . What is T^{-1} ?



Invertible Transformations

Definition

A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **invertible** if there exists another transformation $U: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$T \circ U(x) = x \quad \text{and} \quad U \circ T(x) = x$$

for all x in $\mathbb{R}^n$. In this case we say U is the **inverse** of T , and we write $U = T^{-1}$.

In other words, $T(U(x)) = x$, so T "undoes" U , and likewise U "undoes" T .

Let T = shrinking by a factor of $2/3$ in the plane. What is T^{-1} ?



Fact

If T is an invertible linear transformation with matrix A , then T^{-1} is an invertible linear transformation with matrix A^{-1} .

Fact

A transformation T is invertible if and only if it is both one-to-one and onto.

Some Facts

Say A and B are invertible $n \times n$ matrices.

1. A^{-1} is invertible and its inverse is $(A^{-1})^{-1} = A$.
2. AB is invertible and its inverse is $(AB)^{-1} = A^{-1}B^{-1}$.
3. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Why? $A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$.

You Try It!

Question: If A, B, C are invertible $n \times n$ matrices, what is the inverse of ABC ?

Computing A^{-1}

In general

Let A be an $n \times n$ matrix. Here's how to compute A^{-1} .

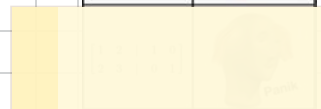
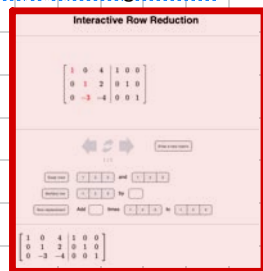
1. Row reduce the augmented matrix $(A \mid I_n)$.
2. If the result has the form $(I_n \mid B)$, then A is invertible and $B = A^{-1}$.
3. Otherwise, A is not invertible.

Example

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array} \right]$$

<https://textbooks.math.gatech.edu/ila/demos/rprinter.html?mat=1,0,4,1,0,0:0,1,2,0,1,0:0,-3,-4,0,0,1&augment=2>



The Invertible Matrix Theorem

A.K.A. The Really Big Theorem of Math 1553

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. The reduced row echelon form of A is the identity matrix I_n .
4. A has n pivots.
5. $Ax = 0$ has no solutions other than the trivial solution.
6. $\text{Nul}(A) = \{0\}$.
7. $\text{nullity}(A) = 0$.
8. The columns of A are linearly independent.
9. The columns of A form a basis for $\mathbb{R}^n$.
10. T is one-to-one.
11. $Ax = b$ is consistent for all b in $\mathbb{R}^n$.
12. $Ax = b$ has a unique solution for each b in $\mathbb{R}^n$.
13. The columns of A span $\mathbb{R}^n$.
14. $\text{Col } A = \mathbb{R}^n$.
15. $\dim \text{Col } A = n$.
16. $\text{rank } A = n$.
17. T is onto.
18. There exists a matrix B such that $AB = I_n$.
19. There exists a matrix B such that $BA = I_n$.

you really have to know these

The Invertible Matrix Theorem

Summary

There are two kinds of **square** matrices:

1. invertible (non-singular), and
2. non-invertible (singular).

For invertible matrices, all statements of the Invertible Matrix Theorem are true.

For non-invertible matrices, all statements of the Invertible Matrix Theorem are false.

The Invertible Matrix Theorem

Example

Question: Is this matrix invertible?

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ -2 & -4 & 1 \end{pmatrix}$$

You Try It!

The Invertible Matrix Theorem

Another Example

Problem: Let A be a 3×3 matrix such that

$$A \begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix} = A \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

True or False: the rank of A is equal to 3.

The Invertible Matrix Theorem

A.K.A. The Really Big Theorem of Math 1553

The Invertible Matrix Theorem

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15. $\dim \text{Col } A = n$.
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17. T is onto.
18. There exists a matrix B such that $AB = I_n$.
19. There exists a matrix B such that $BA = I_n$.

you really have to know these