Section 3.3

Linear Transformations

Linear Transformations
Motivation

In the last two lectures we have been asking questions about transformations, and answering them in the case of matrix transformations.

However, sometimes it is not clear if a transformation is a matrix transformation or not.

Example

For a vector x in $\mathbb{R}^2$, let $T(x)$ be the counterclockwise rotation of x by an angle θ. Is $T(x) = Ax$ for some matrix A?



Linear Transformations

Definition

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if it satisfies the equations for all vectors u, v in $\mathbb{R}^n$ and all scalars c.

(1) T(u+v) = T(u) + T(v) "T respects addition"

(2) T(cv) = c T(v) "T respects scalar multiplication"

↑ T is linear if (1) & (2) hold.

Key observation:

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then

$$A(u+v) = Au + Av \qquad A(cv) = cAv.$$

So if $T(x) = Ax$ is a matrix transformation then,

$$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cv) = cT(v).$$

Some immediate consequences:

Check: if T is linear, then

$$T(0) = 0 \qquad T(cu + dv) = cT(u) + dT(v)$$

for all ~~vectors u, v and scalars~~ c, d. More generally,

$$T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n).$$

In engineering this is called **superposition**.

↑ if you apply a linear transformation to a linear combination of vectors / you get the linear combination of the images w/ same scalar.

Linear Transformations
Dilation

Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = 1.5x$. Is T linear? Check:

↑ need to check (1) T(u+v) = T(u) + T(v) (2) T(cu) = cT(u)

Example

e.g. $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = 1.5\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1.5\\3\end{bmatrix}$

$T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = 1.5\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}1.5a\\1.5b\end{bmatrix}$

$T\left(\begin{bmatrix}c\\d\end{bmatrix}\right) = 1.5\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}1.5c\\1.5d\end{bmatrix}$

$T(u+v) = T\left(\begin{bmatrix}a\\b\end{bmatrix} + \begin{bmatrix}c\\d\end{bmatrix}\right) = T\left(\begin{bmatrix}a+c\\b+d\end{bmatrix}\right) = 1.5\begin{bmatrix}a+c\\b+d\end{bmatrix} = \begin{bmatrix}1.5a+1.5c\\1.5b+1.5d\end{bmatrix}$

$= \begin{bmatrix}1.5a\\1.5b\end{bmatrix} + \begin{bmatrix}1.5c\\1.5d\end{bmatrix}$ (1) ✓

$= T(u) + T(v)$

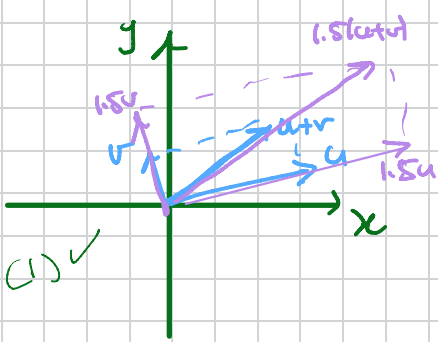
(2) $T\left(k\begin{bmatrix}a\\b\end{bmatrix}\right) = 1.5\,k\begin{bmatrix}a\\b\end{bmatrix}$ (2) ✓

$= k\,1.5\begin{bmatrix}a\\b\end{bmatrix} = kT\left(\begin{bmatrix}a\\b\end{bmatrix}\right)$

Note: T is a matrix transformation!

$$T(x) = \begin{pmatrix}1.5 & 0\\ 0 & 1.5\end{pmatrix}x,$$

as we checked before.

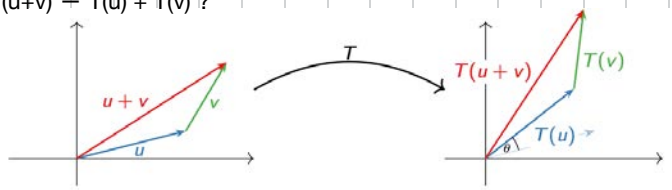


Linear Transformations
Rotation

Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by
$T(x) =$ the vector x rotated counterclockwise by an angle of θ.
Is T linear? Check:

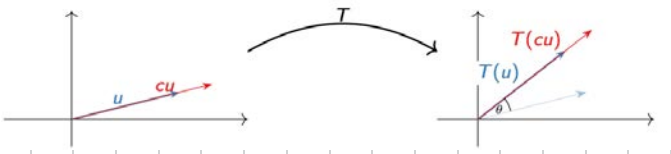
Proof by picture:

(1) $T(u+v) \stackrel{?}{=} T(u) + T(v)$?



add First — then rotate

(2) $T(cv) \stackrel{?}{=} c\,T(v)$?



$T(cu) = cT(u)$

rotate First then scale

Scale first, Then rotate

Linear Transformations
Non-example

Is every transformation a linear transformation?

No! For instance, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$ is not linear.

$T: \mathbb{R}^2 \to \mathbb{R}^3$

Hint: check what happens to T(v) if v=0 the zero vector.

easy way to check if T is linear!

$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sin 0 \\ 0 \cdot 0 \\ \cos 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

So T can't be linear! Why? $T(0) = 0$ ← would have to be true if T was linear

T can't be linear

You Try It!

Poll

Which of the following transformations are linear?

A. $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} |x_1| \\ x_2 \end{pmatrix}$ B. $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_1 - 2x_2 \end{pmatrix}$

C. $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ x_2 \end{pmatrix}$ D. $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + 1 \\ x_1 - 2x_2 \end{pmatrix}$

(D) $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2\cdot 0 + 1 \\ 0 - 2\cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So (D) T is NOT linear.

(B) IS linear but lin hold on the proof

(A) $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$T(u+v) \overset{?}{\neq} T(u) + T(v) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\hookrightarrow T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ this T is NOT linear.

$u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $c = 2$

(C) $T(cu) \overset{?}{\neq} cT(u) = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\hookrightarrow T\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

T is NOT linear.

The Matrix of a Linear Transformation

We will see that a *linear* transformation T is a matrix transformation: $T(x) = Ax$.

But what matrix does T come from? What is A?

Here's how to compute it.

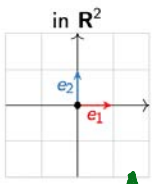
The Matrix of a Linear Transformation

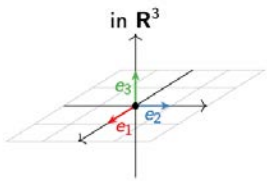
We will see that a *linear* transformation T is a matrix transformation: $T(x) = Ax$.

But what matrix does T come from? What is A?

Here's how to compute it.

How to find $A^?$





First: recall the definitions of e1,e2,...,en

Unit Coordinate Vectors

Definition

The **unit coordinate vectors** in $\mathbb{R}^n$ are

This is what $e_1, e_2, \ldots$ mean, for the rest of the class.

$$e_1 = \begin{pmatrix}1\\0\\\vdots\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\\vdots\\0\\0\end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix}0\\0\\\vdots\\1\\0\end{pmatrix}, \quad e_n = \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}.$$

aka Standard basis vectors in $\mathbb{R}^n$.

in $\mathbb{R}^2$

$e_1 = \begin{bmatrix}1\\0\end{bmatrix}$ $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$

in $\mathbb{R}^3$

$e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

For the matrix A=[1 2 3 ; 4 5 6 ; 7 8 9] compute Ae1, Ae2, and Ae3.

$$Ae_1 = \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\4\\7\end{pmatrix}$$

Ae_1 equals the first column of A.

$$Ae_2 = \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\5\\8\end{pmatrix}$$

Ae_2 is the second column of A.

Note: if A is an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, then $Ae_i = v_i$ for $i = 1, 2, \ldots, n$: multiplying a matrix by e_i gives you the ith column.

$$Ae_3 = \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\6\\9\end{pmatrix}$$

Ae_3 is the third column of A.

sformations are linear?

B. $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2x_1 + x_2\\x_1 - 2x_2\end{pmatrix}$

mmm

What is $A^?$

$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2+0\\1-0\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$ $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0+1\\0-2\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$

$A = \begin{pmatrix}2&1\\1&-2\end{pmatrix}$

First column is $Te_1 = Ae_1$

Second column is $Te_2 = Ae_2$.

Linear Transformations are Matrix Transformations

Theorem

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

This is an $m \times n$ matrix, and T is the matrix transform

The matrix A is called the **standard matrix** for T.

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}2&1\\1&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2x_1 + x_2\\x_1 - 2x_2\end{bmatrix}$$

Same T.

Take-Away

Linear transformations are the same as matrix transformations.

$T(e_1) = Ae_1 =$ first col of A

$T(e_2) = Ae_2 =$ second col of A

$T(e_3) = Ae_3 =$ third col of A

most important info for the day.

Linear Transformations are Matrix Transformations

Example

Before, we defined a **dilation** transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = 1.5x$.
What is its standard matrix?

How to get A? Apply T to $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$ & $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$T(e_1)$ $T(e_2)$

$$A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}$$

Linear Transformations are Matrix Transformations

Example

Question

What is the matrix for the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$T(x) = x$ rotated counterclockwise by an angle θ?

https://www.geogebra.org/graphing/jyzexjma

$T(e_1)$ $T(e_2)$

$$A = \begin{bmatrix} * & * \\ * & * \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{pmatrix} 0.7071067811865 & -0.7071067811865 \\ 0.7071067811865 & 0.7071067811865 \end{pmatrix}$$

$$\begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix}$$

r = 1

θ = 45°

$T(e_2) = \begin{bmatrix} \cos(\theta+90^\circ) \\ \sin(\theta+90^\circ) \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

$(\cos(\theta+90^\circ), \sin(\theta+90^\circ))$

$(\cos\theta, \sin\theta)$

$V = T(e_2)$

$U = T(e_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

θ

e1

e2

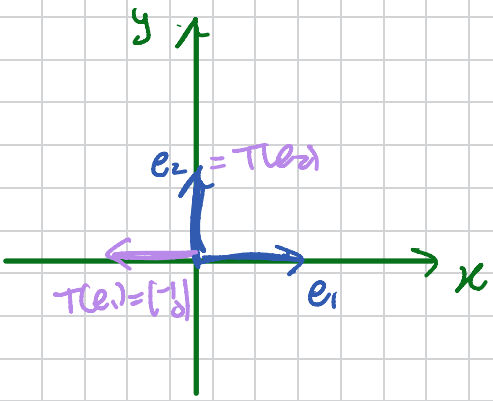
$$A = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

↑ rotation by 45° CCW.

$\cos(\theta+90^\circ) = -\sin(\theta)$

$\frac{\pi}{2}$ π

Ex. Find A for reflection across y-axis in $\mathbb{R}^2$



$$T(e_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x$$

do horizontal reflection to x

Linear Transformations are Matrix Transformations
Example, continued

Question
What is the matrix for the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

(1) Find the standard matrix A of T

(2) is T one-to-one?

(3) is T onto?

SKIP

START HERE

Linear Transformations are Matrix Transformations
Example

Question
Define a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ -y - 5z \end{pmatrix}.$$

What is the standard matrix A for T?

(1) Find the standard matrix A of T $A = [T(e_1)\ T(e_2)\ T(e_3)]$

(2) is T one-to-one? NO

(3) is T onto? yes

$$T(e_1) = T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1+0+0\\-0-0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \leftarrow \text{first col of } A$$

$$T(e_2) = T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0+2+0\\-1-0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} \leftarrow \text{second col of } A$$

$$T(e_3) = T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\-5\end{pmatrix} \leftarrow \text{third col of } A.$$

$$A = \begin{pmatrix}1 & 2 & 3\\0 & -1 & -5\end{pmatrix}. \quad \text{ANS(1)}$$

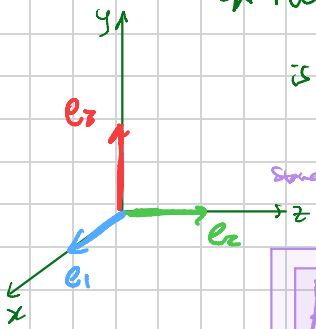
(2) Equivalence: There exist a pivot in every col of A $\iff T(x) = Ax$ 1-1

Not a pivot in every col of A

So T is not 1-1.

(3) Equivalence: There exist a pivot in every row of A $\iff T(x) = Ax$ is onto.

There is a pivot in every row $\Rightarrow$ T is onto.



★ MOST IMPORTANT idea about linear trans. is

$A = [T(e_1)\ T(e_2)\ T(e_3)]$

T e_1, e_2, e_3 ↑ Standard basis vectors

↑ Standard matrix of T

Exam 2

ONE WEEK from toDAY

on Wednesday, Oct 21 @ 6:30pm

Summary

- **Linear transformations** are the transformations that come from matrices.
- The **unit coordinate vectors** $e_1, e_2, \ldots$ are the unit vectors in the positive direction along the coordinate axes.
- You compute the columns of the matrix for a linear transformation by plugging in the unit coordinate vectors.
- This is useful when the transformation is specified geometrically, in terms of a formula, or any other way that isn't as a matrix transformation.

$\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}$ $n < m$ (tall) $T: \mathbb{R}^n \to \mathbb{R}^m$ (eg. $T: \mathbb{R}^2 \to \mathbb{R}^3$)

Then T can NOT be onto

$\begin{bmatrix} * & * \\ * & * \end{bmatrix}$ $n = m$ (square) T is 1-1 $\iff$ T is onto.

$\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$ $n > m$ (wide) T can't be 1-to-1.

Section 3.4

Matrix Multiplication

Motivation

Recall: we can turn any system of linear equations into a matrix equation

$$Ax = b.$$

This notation is suggestive. Can we solve the equation by "dividing by A"?

$$x \overset{??}{=} \frac{b}{A} \quad \text{is} \quad \frac{1}{A}b \; ? \quad \text{is} \quad b \cdot \frac{1}{A} \; ??$$

Answer: Sometimes, but you have to know what you're doing.

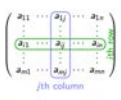
Today we'll study *matrix algebra*: adding and multiplying matrices.

These are not so hard to do. The important thing to understand today is the relationship between *matrix multiplication* and *composition of transformations*.

More Notation for Matrices

Let A be an $m \times n$ matrix.

We write a_{ij} for the entry in the ith row and the jth column. It is called the **ijth entry** of the matrix.



The entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **diagonal entries**; they form the **main diagonal** of the matrix.

A **diagonal matrix** is a square matrix whose only nonzero entries are on the main diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for *all* v in $\mathbf{R}^n$.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a_{ij} entry of A in row i & col j.

a_{ii} anything
$a_{ij} = 0$ $i \neq j$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$a_{11} = 1 \quad a_{12} = 2 \quad a_{13} = 3$

$a_{21} = 4 \quad a_{22} = 5 \quad a_{23} = 6$

row 2, col 2

"acting."

More Notation for Matrices
Continued

The **zero matrix** (of size $m \times n$) is the $m \times n$ matrix 0 with all zero entries.

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \rightsquigarrow \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

A flip

a_{ij} becomes a_{ji}

THIS

$$A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = B$$

a_{13}, a_{22}, b_{22}, b_{13}

first row of A is the first col of A^T etc.

NOT THIS.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T \neq \begin{pmatrix} 4 & 1 \\ 5 & 2 \\ 6 & 3 \end{pmatrix}$$

HARDER WAY?

Matrix Multiplication

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix with columns $v_1, v_2, \ldots, v_p$: (must be equal)

$$B = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

The **product** AB is the $m \times p$ matrix with columns $Av_1, Av_2, \ldots, Av_p$:

$$AB \overset{\text{def}}{=} \begin{pmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_p \\ | & | & & | \end{pmatrix}.$$

(The equality is a definition)

In order for $Av_1, Av_2, \ldots, Av_p$ to make sense, the number of columns of A has to be the same as the number of rows of B. Note the sizes of the product!

Example

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 14 & -10 \\ 32 & -28 \end{pmatrix}$$

(2×3, 3×2, 2×3, 3×1, 2×2; first col of B, second col of B; A, B)

The Row-Column Rule for Matrix Multiplication

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1+2\cdot 2+3\cdot 3 & \square \\ \square & \square \end{pmatrix} = \begin{pmatrix} 14 & \square \\ \square & \square \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \square & \square \\ 4\cdot 1+5\cdot 2+6\cdot 3 & \square \end{pmatrix} = \begin{pmatrix} \square & \square \\ 32 & \square \end{pmatrix}$$

C_{11} entry is first row of A times first col of B

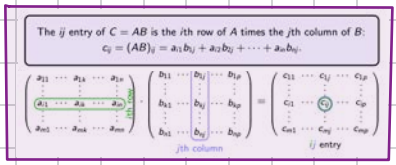
$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 14 & -10 \\ 32 & -28 \end{bmatrix} = C$$

(2×3, 3×2, 2×2)

C_{21} entry is second row of A times first col of B

The ij entry of $C = AB$ is the ith row of A times the jth column of B:

$$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

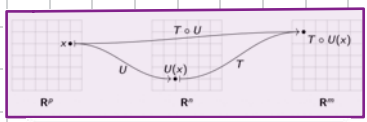


$$f \circ g(x) = f(g(x))$$

"f circle g"

f composed w g.

f after g.



The matrix of the composition is the product of the matrices!

Composition of Transformations

Definition

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ and $U: \mathbb{R}^p \to \mathbb{R}^n$ be transformations. The **composition** is the transformation

$$T \circ U: \mathbb{R}^p \to \mathbb{R}^m \quad \text{defined by} \quad T \circ U(x) = T(U(x)).$$

Example: Find the standard matrix for the transformation T ∘U where
U is the transformation which rotates vectors in R^2 by 90^deg *clockwise*, and (go first)
T is the transformation which projects vectors in R^2 to the x-axis. (go second)

Step 1 $U(x) = Ax$ A standard matrix of rotation U

$T(x) = Bx$ B Standard matrix of projection T.

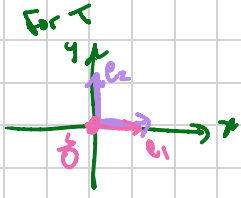
For U



$$A = \begin{bmatrix} U(e_1) & U(e_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{(first)}$$

$$B = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{(second)}$$

For T



$T \circ U(x) = AB\,x$? ↔ which one?

$T \circ U(x) = BA\,x$?

$$T \circ U(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$

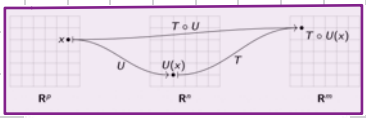
composition's standard matrix.

Composition of Transformations

Definition

Let $T: \mathbf{R}^n \to \mathbf{R}^m$ and $U: \mathbf{R}^p \to \mathbf{R}^n$ be transformations. The **composition** is the transformation

$$T \circ U: \mathbf{R}^p \to \mathbf{R}^m \quad \text{defined by} \quad T \circ U(x) = T(U(x)).$$



The matrix of the composition is the product of the matrices!

Example: Find the standard matrix for the transformation T ∘U where
U is the transformation which rotates vectors in R^2 by 90^deg *clockwise*, and
T is the transformation which projects vectors in R^2 to the x-axis.

Composition of Linear Transformations

Another Example

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by 45°, and let $U: \mathbb{R}^2 \to \mathbb{R}^2$ scale the x-coordinate by 1.5. Let's compute their standard matrices A and B.

Also: find the standard matrix for G=T ∘U

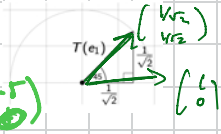
$$T \circ U(x) = \begin{pmatrix} 3/2\sqrt{2} & -1/\sqrt{2} \\ 3/2\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

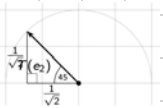
(first col)

$$T \circ U(e_1) = \begin{pmatrix} 3/2\sqrt{2} \\ 3/2\sqrt{2} \end{pmatrix}$$

$$U(e_1) = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



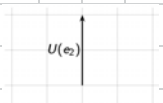


(second col)

$$\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = T \circ U(e_2)$$

$$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = U(e_2)$$





Other method

$$U(x) = Ax$$
$$T(x) = Bx$$

$$U(e_1) = U\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} \quad U(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T(e_1) = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad T(e_2) = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$T \circ U$ matrix is $BA = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

$$= \begin{bmatrix} (1.5)(1/\sqrt{2}) & -1/\sqrt{2} \\ (1.5)(1/\sqrt{2}) & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3/\sqrt{2} & -1/\sqrt{2} \\ 3/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

You Try It!

Poll

Poll

Do there exist *nonzero* matrices A and B with $AB = 0$?

Hint: think geometrically!

Idea: project to x-axis then project to y-axis?

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

EXAM 2 THIS WEEK on Wed 10/21 @ 6:30 pm

Covers up to §3.4 including TODAY'S content (last not §3.5/3.6 on test? invertibility)

You Try It!

Composition of Linear Transformations
Example

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ and $U: \mathbb{R}^2 \to \mathbb{R}^3$ be the matrix transformations

$$T(x) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} x \qquad U(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} x.$$

A B

(1) Compute AB and BA

(2) which one is the standard matrix for "First T Then U"?

Example: What is the standard matrix for the transformation which first applies the transformation T to a vector x and

$$AB = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

2×3 (matches) 3×2 → 2×2

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

3×2 (matches) 2×3 → 3×3 ✓

$U \circ T$ "U after T"

$T: \mathbb{R}^3 \to \mathbb{R}^2$ Then $U: \mathbb{R}^2 \to \mathbb{R}^3$

input of T size 3 (domain of T is $\mathbb{R}^3$)

output of U is size 3 (= domain of U is $\mathbb{R}^3$)

Addition and Scalar Multiplication for Matrices

You add two matrices component by component, like with vectors.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix}$$

Note you can only add two matrices *of the same size*.

You multiply a matrix by a scalar by multiplying each component, like with vectors.

$$c\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{pmatrix}.$$

These satisfy the expected rules, like with vectors:

$$A+B = B+A \qquad (A+B)+C = A+(B+C)$$
$$c(A+B) = cA+cB \qquad (c+d)A = cA+dA$$
$$(cd)A = c(dA) \qquad A+0 = A$$

Rules that work. (about addition & scalar mult of matrices)

Properties of Matrix Multiplication

Mostly matrix multiplication works like you'd expect. Suppose A has size $m \times n$, and that the other matrices below have the right size to make multiplication work.

$$A(BC) = (AB)C \qquad A(B+C) = (AB+AC)$$
$$(B+C)A = BA+CA \qquad c(AB) = (cA)B$$
$$c(AB) = A(cB) \qquad I_m A = A$$
$$AI_n = A$$

More rules that work (about matrix multiplication)

$ABC = A(BC)$ = A(BC) compute first

$ABC = (AB)C$ = (AB)C compute first, then multiply by C

$AB \neq BA$

Properties of Matrix Multiplication
Caveats

NON-RULES

$AB \neq BA$ in general

Warnings!
- AB is usually not equal to BA.

$$\underset{A}{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}}\underset{B}{\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}} = \begin{bmatrix}0 & -1\\ 2 & 0\end{bmatrix}$$

$AB \neq BA$

$$\underset{B}{\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}}\underset{A}{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}} = \begin{bmatrix}0 & -2\\ 1 & 0\end{bmatrix}$$

A rotation by 90° CCW



B scaling of x-direction by 2 (horizontal expansion)

"order matters when doing TOU or UoT"

NON-RULE $AB = AC \not\Rightarrow B = C$ or $A = 0$

Warnings!
- $AB = AC$ does not imply $B = C$, even if $A \neq 0$.

$$\underset{A}{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}}\underset{B}{\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}} = \begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}$$

(number $ax = ay$ and $a \neq 0$ $\Rightarrow x = y$ ✓)

So $AB = AC$ but $B \neq C$ and $A \neq 0$.

$$\underset{A}{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}}\underset{C}{\begin{pmatrix}1 & 2\\ 5 & 6\end{pmatrix}} = \begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}$$

non-rule $AB = 0 \not\Rightarrow A = 0$ or $B = 0$

Warnings!
- $AB = 0$ does not imply $A = 0$ or $B = 0$.

$$\underset{A}{\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}}\underset{B}{\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

So $AB = 0$ but $A \neq 0$ $B \neq 0$.

Powers of a Matrix

Definition

Let n be a positive whole number and let A be a square matrix. The **nth power** of A is the product

$$A^n = \underbrace{A \cdot A \cdot \cdots \cdot A}_{n \text{ times}}$$

Example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A^2 = A*A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = A*A*A = A*A^2 = A^2*A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$A^n = \underbrace{A*A*\cdots*A}_{n \text{ times}} = ?? \quad ? \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ??$$

Summary

- The product of an $m \times n$ matrix and an $n \times p$ matrix is an $m \times p$ matrix. I showed you two ways of computing the product.
- Composition of linear transformations corresponds to multiplication of matrices.
- You have to be careful when multiplying matrices together, because things like commutativity and cancellation fail.
- You can take powers of square matrices.

have to match

$m \times n$ $\quad n \times p$

AB

result is $m \times p$

Standard matrix of T

standard matrix of U

$$T \circ U(x) = ABx$$

U goes first then T.

In the future...

Solve for X matrix $\quad A, B, X, I$ are all $n \times n$ matrices

$$A + BX = I$$

Solve. $A + BX = I \Rightarrow B^{-1}BX = B^{-1}(I - A)$

$\Rightarrow X \overset{??}{=} B^{-1}(I - A)$ or $(I - A)B^{-1}$

§3.5 & 3.6 on inverses & IMT are NOT covered on Wednesday's exam.

Section 3.5 and 3.6

Matrix Inverses and the Invertible Matrix Theorem

The Definition of Inverse

Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that

$AB = I_n$ and $BA = I_n$, where $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ (identity matrix)

In this case, B is the **inverse** of A, and is written A^{-1}.

Example $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

I claim $B = A^{-1}$. Check:

Compare to numbers.

3 is invertible (using multiplication) b/c $3 * \frac{1}{3} = 1$.

↑ multiplicative identity

$a * 1 = a$.

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

?? check

I_2 identity matrix of size 2×2

So A & B are inverses and $A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

$$A * A^{-1} = I.$$

Sanity check

$$BA = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ ✓

You Try It!

Poll

Do there exist two matrices A and B such that AB is the identity, but BA is not? If so, find an example. (Both products have to make sense.)

Hint: don't assume A and B have the same size.

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ not the 3×3 identity matrix.

IF A & B both square $n \times n$ with same size.

However

If A and B are *square* matrices, then

$AB = I_n$ if and only if $BA = I_n$.

So in this case you only have to check one.

Key observation: Having an inverse of a coefficient matrix of a system Ax=b makes solving the system very easy!

Solving Linear Systems via Inverses

Example

Example

Solve the system

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 1 \\ x_1 \quad\quad + 3x_3 &= 1 \\ 2x_1 + 2x_2 + 3x_3 &= 1 \end{aligned} \quad \text{using} \quad \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}.$$

Solve $Ax = b$

$$A = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x = A^{-1}b = \begin{bmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Check soln?

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 1 \quad \checkmark \quad -4+3+2 = 1 \checkmark \\ x_1 \quad\quad + 3x_3 &= 1 \quad \checkmark \quad -2 + 3 = 1 \\ 2x_1 + 2x_2 + 3x_3 &= 1 \quad \checkmark \quad -4+2+3 = 1 \end{aligned}$$

Old way: (want Soln)

$$[A|b] \sim \cdots \sim [I|\ *\ ]$$

New way: Solve

$$Ax = b$$

$$\Rightarrow A^{-1}Ax = A^{-1}b$$

$$\Rightarrow Ix = A^{-1}b$$

$$\Rightarrow \boxed{X = A^{-1}b}$$

The advantage of using inverses is it doesn't matter what's on the right-hand side of the $=$:

$$\begin{cases} 2x_1 + 3x_2 + 2x_3 = b_1 \\ x_1 \quad\quad + 3x_3 = b_2 \\ 2x_1 + 2x_2 + 3x_3 = b_3 \end{cases} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= \begin{pmatrix} -6b_1 - 5b_2 + 9b_3 \\ 3b_1 + 2b_2 - 4b_3 \\ 2b_1 + 2b_2 - 3b_3 \end{pmatrix}$$

Computing A^{-1}
The 2 × 2 case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **determinant** of A is the number

$$\det(A) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Facts:

1. If $\det(A) \neq 0$, then A is invertible and $A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
2. If $\det(A) = 0$, then A is not invertible.

↑ is zero??

Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{4-6}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{-1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

for $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1*4 - 2*3$$
$$= 4 - 6 = -2$$

So the determinant of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is -2.

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Check

$$AA^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} = \begin{pmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ ✓

Invertible Linear Transformations
Examples

Let T = counterclockwise rotation in the plane by 45°. Its matrix is

$$A = \begin{pmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then T^{-1} = counterclockwise rotation by −45°. Its matrix is what matrix?

Let T = counterclockwise rotation in the plane by 45°. What is T^{-1}?



$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

CCW rotation by 45°

So $A^{-1} = \frac{1}{\frac{1}{2} - \left(-\frac{1}{2}\right)}\begin{bmatrix} 1/\sqrt{2} & +1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Clockwise rotation by 45°.

EXAM TODAY at 6:30 pm !!!

NO PHONES!

Invertible Transformations

Definition

A transformation $T: \mathbf{R}^n \to \mathbf{R}^n$ is **invertible** if there exists another transformation $U: \mathbf{R}^n \to \mathbf{R}^n$ such that ← First T then U

$$T \circ U(x) = x \quad \text{and} \quad U \circ T(x) = x$$

for all x in $\mathbf{R}^n$. In this case we say U is the **inverse** of T, and we write $U = T^{-1}$.

In other words, $T(U(x)) = x$, so T "undoes" U, and likewise U "undoes" T.

$T(x) = Ax$
$U(x) = Bx$

Then If T & U are inverses
Then $AB = BA = I$
so A & B are inverses.

Let T = shrinking by a factor of 2/3 in the plane. What is T^{-1}?



Fact

If T is an invertible linear transformation with matrix A, then T^{-1} is an invertible linear transformation with matrix A^{-1}.

Fact

A transformation T is invertible if and only if it is both one-to-one and onto.

Some Facts

Say A and B are invertible $n \times n$ matrices.

1. A^{-1} is invertible and its inverse is $(A^{-1})^{-1} = A$.
2. AB is invertible and its inverse is $(AB)^{-1} = A^{-1}B^{-1}$.
   Why? $(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$.
3. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.
   Why? $A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$.

$(A^T)^{-1} = (A^{-1})^T$

$(AB)^{-1} \neq A^{-1}B^{-1}$

Why not?

$ABA^{-1}B^{-1} \stackrel{??}{=} I$??

Not allowed

$(AB)(AB)^{-1} = ABB^{-1}A^{-1} = AIA^{-1}$

$(AB)^{-1}$

$= AA^{-1}$

$= I.$

You Try It!

Question: If A, B, C are invertible $n \times n$ matrices, what is the inverse of ABC?

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$?

Check $ABCC^{-1}B^{-1}A^{-1} = I$

A B

(Socks then Shoes)$^{-1}$

B^{-1}

= take off Shoes then take off Socks

A^{-1}

Computing A^{-1}
In general

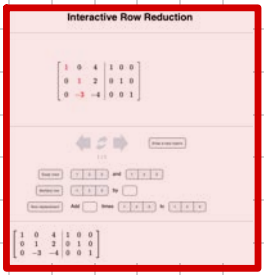
Let A be an $n \times n$ matrix. Here's how to compute A^{-1}.

1. Row reduce the augmented matrix $(A \mid I_n)$.
2. If the result has the form $(I_n \mid B)$, then A is invertible and $B = A^{-1}$.
3. Otherwise, A is not invertible.

Example

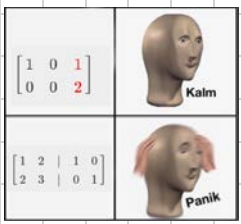
$$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}$$

https://textbooks.math.gatech.edu/ila/demos/rrinter.html?mat=1,0,4,1,0,0:0,1,2,0,1,0:0,-3,-4,0,0,1&augment=2



$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{bmatrix}$



Check $AA^{-1} \stackrel{?}{=} I$?

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} \begin{pmatrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

META $[A|I] \sim \cdots \sim [I|A^{-1}]$

row reduce a big augmented matrix to RREF.

The Invertible Matrix Theorem
A.K.A. The Really Big Theorem of Math 1553

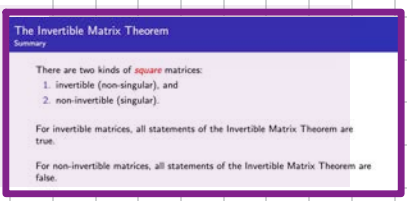
1553 "Map" of concepts.

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T: \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent. ← all these (1.)–(19.) are ALL TRUE or ALL FALSE for any given square matrix A.

1. A is invertible.
2. T is invertible.
3. The reduced row echelon form of A is the identity matrix I_n.
4. A has n pivots.
5. $Ax = 0$ has no solutions other than the trivial solution.
6. $\text{Nul}(A) = \{0\}$.
7. $\text{nullity}(A) = 0$.
8. The columns of A are linearly independent.
9. The columns of A form a basis for $\mathbf{R}^n$.
10. T is one-to-one.
11. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
12. $Ax = b$ has a unique solution for each b in $\mathbf{R}^n$.
13. The columns of A span $\mathbf{R}^n$.
14. $\text{Col}\, A = \mathbf{R}^n$.
15. $\dim \text{Col}\, A = n$.
16. $\text{rank}\, A = n$.
17. T is onto.
18. There exists a matrix B such that $AB = I_n$.
19. There exists a matrix B such that $BA = I_n$.

you really have to know these



When A is invertible we say A is NON-SINGULAR.

if A is not invertible we say A is SINGULAR

$$A = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

v_1 v_2 v_3

What's the most likely

most likely $\dim \text{Span}\{v_1\} = 1$

most likely $\dim \text{Span}\{v_1, v_2\} = 2$

most likely $\dim \text{Span}\{v_1, v_2, v_3\} = 3$

The Invertible Matrix Theorem
Example

Question: Is this matrix invertible?

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ -2 & -4 & 1 \end{pmatrix}$$

No $A * \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = -2\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix} + 0\begin{pmatrix} -1 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

So Nul A has a non-zero vector
So A has a free variable
it doesn't have 3 pivots
& A is SINGULAR (not invertible)

You Try It!

The Invertible Matrix Theorem
Another Example

Problem: Let A be a 3×3 matrix such that

$$A\begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix} = A\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

True or False: the rank of A is equal to 3.

$T(x) = Ax$

Then $T\left(\begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix}\right) = T\left(\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}\right)$

but inputs are different,
So T is many-to-one
(So T is Not 1-to-1.)

A is not invertible
rank of A is NOT 3 (Col A is not dimension 3)

The Invertible Matrix Theorem
A.K.A. The Really Big Theorem of Math 1553

The Invertible Matrix Theorem
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you might be able to show these