Chapter 5

Eigenvalues and Eigenvectors

Section 5.1

Eigenvalues and Eigenvectors

Review What is it???

* determinant
* transformation
* vector
* subspace
* pivot
* matrix
* eigenvalue ???

Definition
If A is an $n \times n$ square matrix and there is a nonzero vector v in $\mathbf{R}^n$ and a scalar λ in $\mathbf{R}$ with

$$Av = \lambda v,$$

then v is an **eigenvector** of A with **eigenvalue** λ.

{ x eigenvector, λ eigenvalue

$Ax = \lambda x$ $x \neq 0$

Verifying Eigenvectors

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

A v

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} = \begin{bmatrix} 0+24+8 \\ 8+0+0 \\ 0+2+0 \end{bmatrix} = \begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}$$

So $\begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}$ is an eigenvector of A w/ eigenvalue 2.

Note: Eigenvectors are by definition nonzero. Eigenvalues may be equal to zero.

Note: This is the most important definition in the course.

Example

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$Av = \begin{bmatrix} 2 & 2 \\ -4 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ So $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvalue of A w/ eigenvalue $\lambda = 4$.

You Try It!

Poll

Poll

Which of the vectors

A. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ C. $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ D. $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ E. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

are eigenvectors of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$?

What are the eigenvalues?

Definition
If A is an $n \times n$ square matrix and there is a nonzero vector v in $\mathbf{R}^n$ and a scalar λ in $\mathbf{R}$ with

$$Av = \lambda v,$$

then v is an **eigenvector** of A with **eigenvalue** λ.

{ x eigenvector, λ eigenvalue

$Ax = \lambda x$ $x \neq 0$

$A\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \lambda\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

NO $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is NOT an eigenvector of A.

(A) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ So yes $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ e-vector w/ e-value $\lambda = 2$

(B) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ So yes $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ eigenvector w/ eigenvalue $\lambda = 0$.

(C) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ So yes $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ <— also. (also $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is in Nul(A))

(D) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ So $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ NOT an eigenvector.

Verifying Eigenvalues

Question: Is $\lambda = 3$ an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$?

Q: is there a **nonzero** vector x such that Ax=3x?

yes. e.g. $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

Check $A\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -8-4 \\ 4-1 \end{bmatrix} = \begin{bmatrix} -12 \\ 3 \end{bmatrix} = 3\begin{bmatrix} -4 \\ 1 \end{bmatrix}$ ✓

Solve for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} 2x_1 - 4x_2 = 3x_1 \\ -x_1 - x_2 = 3x_2 \end{cases}$$

$$\Rightarrow \quad \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 3\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2x_1 - 3x_1 - 4x_2 &= 0 \\ -x_1 - x_2 - 3x_2 &= 0 \end{aligned}$$

$$\Rightarrow \quad \left(\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} (2-3)x_1 - 4x_2 &= 0 \\ -x_2 + (-1-3)x_2 &= 0 \end{aligned}$$

$$\begin{bmatrix} 2-3 & -4 \\ -1 & -1-3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

compute the Null space of $\begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix}$

↑ row reduce this.

Solve $Ax = \lambda x$

$\Rightarrow Ax - \lambda x = 0$

$\Rightarrow (A - \lambda I)x = 0$

row reduce this matrix

$\lambda = 3$

$$A - 3I = \begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix} \sim \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix} \quad x_2$$

$x_1 = -4x_2$

$x_2 = x_2$ (Eqn 0)

$$x = x_2\begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

---

also $\begin{bmatrix} -8 \\ 2 \end{bmatrix}$ is also an eigenvector w/ eigenvalue $\lambda = 3$??

$$\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -8 \\ 2 \end{bmatrix} = \begin{bmatrix} -16 + (-8) \\ 8 - 2 \end{bmatrix} = \begin{bmatrix} -24 \\ 6 \end{bmatrix}$$

https://textbooks.math.gatech.edu/ila/demos/rabbits/book_demo.html

A Biology Question

Motivation

In a population of rabbits:

1. half of the newborn rabbits survive their first year;
2. of those, half survive their second year;
3. their maximum life span is three years;
4. rabbits have 0, 6, 8 baby rabbits in their three years, respectively.

If you know the population one year, what is the population the next year?

f_n = first-year rabbits in year n
s_n = second-year rabbits in year n
t_n = third-year rabbits in year n

The rules say:

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}.$$

Let $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ and $v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$. Then $Av_n = v_{n+1}$. ← difference equation

A Biology Question

Continued

If you know v_0, what is v_{10}?

$$v_{10} = Av_9 = AAv_8 = \cdots = A^{10}v_0.$$

This makes it easy to compute examples by computer: [interactive]

v_0	v_{10}	v_{11}
$\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}$	$\begin{pmatrix} 30189 \\ 7761 \\ 1844 \end{pmatrix}$	$\begin{pmatrix} 61316 \\ 15095 \\ 3881 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 9459 \\ 2434 \\ 577 \end{pmatrix}$	$\begin{pmatrix} 19222 \\ 4729 \\ 1217 \end{pmatrix}$
$\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$	$\begin{pmatrix} 28856 \\ 7405 \\ 1765 \end{pmatrix}$	$\begin{pmatrix} 58550 \\ 14428 \\ 3703 \end{pmatrix}$

What do you notice about these numbers?

1. Eventually, each segment of the population doubles every year: $Av_n = v_{n+1} = 2v_n$.
2. The ratios get close to $(16 : 4 : 1)$:

$$v_n = (\text{scalar}) \cdot \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}.$$



Q: Find $X \neq 0$ w/ eigenvalue $\lambda = 8$

IDEA: $A - 8I$ row reduce

IMT DLC.

Problem: One of the eigenvalues of A is $\lambda = 8$. Identify the other eigenvalue and find at least two eigenvectors for each eigenvalue for the matrix $A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$

Important

If x is a nonzero vector in $\text{Nul}(A)$, then x is an eigenvector of A with eigenvalue $\lambda = 0$.

Consequently, non-invertible matrices always have $\lambda = 0$ as an eigenvalue.

Q: A is invertible or singular?

by defn of singular, A is NOT invertible

by IMT that means

$Ax = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for some $x \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So $Ax = 0x$ for some $x \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So A has eigenvalue $\lambda = 0$.

So for example $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ (x_2) $\quad x = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Check $\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ ✓

$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of A w/ eigenvalue $\lambda = 0$.

Q: eigenvector for $\lambda = 8$ for $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$

$A - 8I = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = \begin{pmatrix} -6 & 4 \\ 3 & -2 \end{pmatrix} \sim \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2/3 \\ 0 & 0 \end{bmatrix}$ (x_2)

$x = x_2 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$ is an eigenvector w/ eigenvalue $\lambda = 8$.

Check $\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = \begin{pmatrix} 4/3 + 4 \\ 2 + 6 \end{pmatrix} = \begin{pmatrix} 16/3 \\ 8 \end{pmatrix} = 8 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$ ✓

Sanity check is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ an eigenvector of A? w/ $\lambda = 8$ ($x_2 = 3$)

$(A - 8I)x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow Ax - 8Ix = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow Ax = 8x$ ✓

Warning: T/F If x eigenvector of A w/ $\lambda = 8$ then 3x eigenvector of A w/ $\lambda = 24$

Eigenspaces

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The **λ-eigenspace** of A is the set of all eigenvectors of A with eigenvalue λ, plus the zero vector:

$$\begin{aligned}\lambda\text{-eigenspace} &= \{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\} \\ &= \{v \text{ in } \mathbf{R}^n \mid (A - \lambda I)v = 0\} \\ &= \mathrm{Nul}(A - \lambda I).\end{aligned}$$

$A - \lambda I$ invertible??

$\det(A - \lambda I) = 0$ or not?

Since the λ-eigenspace is a null space, it is a *subspace* of $\mathbf{R}^n$.

How do you find a basis for the λ-eigenspace? Parametric vector form!

$Ax = \lambda x$ $x \neq 0$ soln?

$\Leftrightarrow (A - \lambda I)x = 0$ $x \neq 0$

$\Leftrightarrow$ $A - \lambda I$ not invertible.

Eigenspaces
Example

Find a basis for the 3-eigenspace of

$$A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}.$$

Solve $(A - 3I)x = 0$ & get parametric vector form

$$A - 3I = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$$

x_2

$x_1 + 4x_2 = 0$

$x_2 = x_2$ free

So A has 3 as an eigenvalue

$A - 3I$ not invertible

$$x = x_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

So $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$ is an eigenvector along any non-zero scalar mult.

So the 3-eigenspace is $\mathrm{Span}\left\{\begin{bmatrix} -4 \\ 1 \end{bmatrix}\right\}$ is a line in $\mathbb{R}^2$

The eigenspace has all the 3-eigenvectors AND $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenspaces
Example

Find a basis for the 2-eigenspace of

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

META:

1. Compute $A - \lambda I$
2. row reduce to RREF & get parametric vector form
3. extract the vectors to get a basis for $\mathrm{Nul}(A - \lambda I)$.

① $A - 2I = \begin{bmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

wants a basis for $\mathrm{Nul}(A - 2I)$

$$= \begin{bmatrix} 3/2 & 0 & 3 \\ -3/2 & 0 & -3 \\ -3/2 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 x_3

$x_1 + 0x_2 + 2x_3 = 0$

$x_2 = x_2$

$x_3 = x_3$

$$x = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

if the question asks for a basis then DON'T write SPAN.

So a basis for the eigenspace is ~~Span~~ $\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\}$

The 2-eigenspace is $\mathrm{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\}$

Question: how many eigenvalues can a matrix have? Can they have more than one eigenvalue?

Eigenspaces
Example

Find a basis for the $\frac{1}{2}$-eigenspace of

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

$(A - \frac{1}{2}I)x = 0$ Solns?

$$A - \tfrac{1}{2}I = \begin{bmatrix} 7/2 - 1/2 & 0 & 3 \\ -3/2 & 2 - 1/2 & -3 \\ -3/2 & 0 & -1 - 1/2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ -3/2 & 3/2 & -3 \\ -3/2 & 0 & -3/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ -3 & 3 & -6 \\ -3 & 0 & -3 \end{bmatrix}$$

$$\sim 3R_1 + R_2 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad x_3$$

$x_1 + x_3 = 0$
$x_2 - x_3 = 0$
$x_3 = x_3$ (free)

$$X = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Check $Ax = \frac{1}{2}X$ is $X = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ ($x_3 = 1$) really an eigenvector?

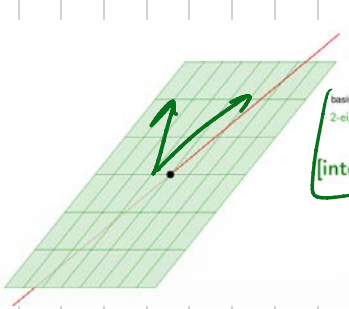
$$\begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -7/2 + 3 \\ 3/2 + 2 - 3 \\ 3/2 - 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$ ✓

https://textbooks.math.gatech.edu/ila/demos/eigenspace.html?nomult

Eigenspaces
Example: picture

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

We computed bases for the 2-eigenspace and the 1/2-eigenspace:



basis for the $\frac{1}{2}$-eigenspace: $\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}$

basis for the 2-eigenspace: $\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right\}$

[interactive]

$\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right\}$

Ins and outs

Same eigenvalue.

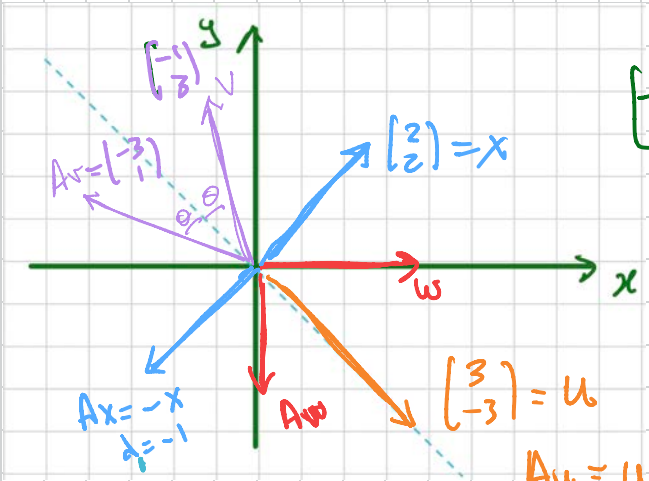
Find eigenvalues/eigenvectors by observation.

Eigenspaces
Geometry; example

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

https://textbooks.math.gatech.edu/ila/demos/eigenspace.html?mat=0,-1:-1,0&nomult



$$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}-3\\1\end{bmatrix} \neq \lambda\begin{bmatrix}-1\\3\end{bmatrix}$$

$Aw = u$

$\lambda_2 = 1$

$$[T(e_1)\ T(e_2)] = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$$

$\lambda_1 = -1 \qquad d_2 = 1$

Check ✓

$$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}-2\\-2\end{bmatrix}$$ ✓

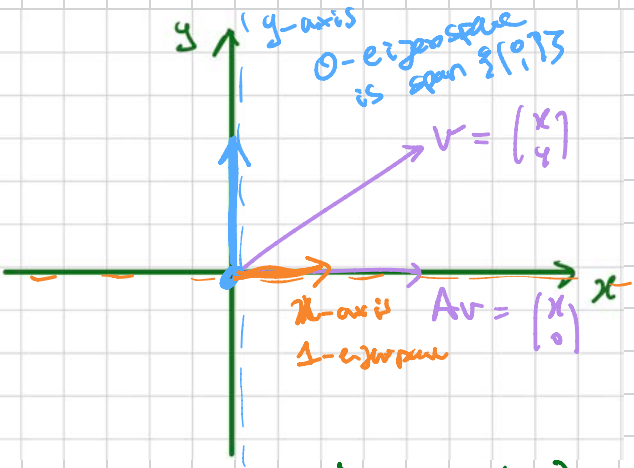
$$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\begin{bmatrix}3\\-3\end{bmatrix} = \begin{bmatrix}3\\-3\end{bmatrix}$$ ✓

Eigenspaces
Geometry; example

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the vertical projection onto the x-axis, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

https://textbooks.math.gatech.edu/ila/demos/eigenspace.html?mat=1,0:0,0&nomult



$\lambda_1 = 0$

$Ax = 0x$

$\lambda_2 = 1$

$Ax = 1x$

$$A = [T(e_1)\ T(e_2)] = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\0\end{pmatrix} \qquad \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}, \quad \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

DLC.

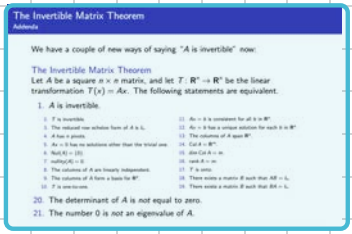
A Matrix is Invertible if and only if Zero is not an Eigenvalue

Fact: A is invertible if and only if 0 is not an eigenvalue of A.

Why?

0 is an eigenvalue of $A \iff Ax = 0x$ has a nontrivial solution (def of eigenvector)
$\iff Ax = 0$ has a nontrivial solution
$\iff A$ is not invertible. (IMT).

invertible matrix theorem



The Invertible Matrix Theorem

Addenda

We have a couple of new ways of saying "A is invertible" now:

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix, and let $T: \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. The reduced row echelon form of A is I_n.
4. A has n pivots.
5. $Ax = 0$ has no solutions other than the trivial one.
6. $\mathrm{Null}(A) = \{0\}$.
7. $\mathrm{nullity}(A) = 0$.
8. The columns of A are linearly independent.
9. The columns of A form a basis for $\mathbb{R}^n$.
10. T is one-to-one.
11. $Ax = b$ is consistent for all b in $\mathbb{R}^n$.
12. $Ax = b$ has a unique solution for each b in $\mathbb{R}^n$.
13. The columns of A span $\mathbb{R}^n$.
14. $\mathrm{Col}\, A = \mathbb{R}^n$.
15. $\dim \mathrm{Col}\, A = n$.
16. $\mathrm{rank}\, A = n$.
17. T is onto.
18. There exists a matrix B such that $AB = I_n$.
19. There exists a matrix B such that $BA = I_n$.
20. The determinant of A is not equal to zero.
21. The number 0 is not an eigenvalue of A.

another way

$\mathrm{Nul}(A - \lambda I)$

λ-eigenspace

if $\lambda = 0$

$\mathrm{Nul}(A - 0I) = \mathrm{Nul}\, A$

$Ax = 0x$ for some $x \neq 0$

$\iff \lambda = 0$ is an eigenvalue of A

$\iff x$ is nonzero in $\mathrm{Nul}(A) \iff A$ not invertible.

Eigenvectors with Distinct Eigenvalues are Linearly Independent

Fact: If $v_1, v_2, \ldots, v_k$ are eigenvectors of A with *distinct* eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Warning! Knowing vectors v1 and v2 have the **same** eigenvalue does NOT mean that they are linearly dependent!

See example w/ 2-dimensional eigenspace.

You Try It!

Poll

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise rotation by 45°, and let A be the matrix for T.



Poll

Find an eigenvector of A without doing any computations.

A. Okay. **B.** No way.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

If $\theta = 45°$

then what eigenvalues does A have?

$$A = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

has no (real) eigenvalues.

Warning!

λ could be complex!!

(not until §5.5)

Eigenspaces

Summary

Let A be an $n \times n$ matrix and let λ be a number.

1. λ is an eigenvalue of A if and only if $(A - \lambda I)x = 0$ has a nontrivial solution, if and only if $\mathrm{Nul}(A - \lambda I) \neq \{0\}$.
2. In this case, finding a basis for the λ-eigenspace of A means finding a basis for $\mathrm{Nul}(A - \lambda I)$ as usual, i.e. by finding the parametric vector form for the general solution to $(A - \lambda I)x = 0$.
3. The eigenvectors with eigenvalue λ are the nonzero elements of $\mathrm{Nul}(A - \lambda I)$, i.e. the nontrivial solutions to $(A - \lambda I)x = 0$.

Summary

- **Eigenvectors** and **eigenvalues** are the most important concepts in this course.
- Eigenvectors are by definition nonzero; eigenvalues may be zero.
- The eigenvalues of a triangular matrix are the diagonal entries.
- A matrix is invertible if and only if zero is not an eigenvalue.
- Eigenvectors with distinct eigenvalues are linearly independent.
- The λ-eigenspace is the set of all λ-eigenvectors, plus the zero vector.
- You can compute a basis for the λ-eigenspace by finding the parametric vector form of the solutions of $(A - \lambda I_n)x = 0$.

next section!

Q: How to find the λ's?

Section 5.2

The Characteristic Polynomial

The Characteristic Polynomial

Definition

Let A be a square matrix. The **characteristic polynomial** of A is

$$f(\lambda) = \det(A - \lambda I).$$

The **characteristic equation** of A is the equation

$$f(\lambda) = \det(A - \lambda I) = 0.$$

The Characteristic Polynomial

Example

Question: What are the eigenvalues of

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}?$$

Important

The eigenvalues of A are the roots of the characteristic polynomial $f(\lambda) = \det(A - \lambda I)$.

Key observation

$(A-\lambda I)x = 0$ has a nonzero soln for x.

$\Leftrightarrow$ $A - \lambda I$ not invertible

$\Leftrightarrow$ $\det(A-\lambda I) = 0$.

Step 1: compute $\det(A-\lambda I) = p(\lambda)$ the characteristic polynomial.

Step 2: set $p(\lambda) = 0$ & solve for λ

When is $A-\lambda I$ not invertible? $\Leftrightarrow$ for what λ is $\det(A-\lambda I) = 0$?

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = (5-\lambda)(1-\lambda) - 2*2$$

$$= 5 - \lambda - 5\lambda + \lambda^2 - 4 = \lambda^2 - 6\lambda + 1 = 0$$

Solve this for λ

$$\lambda = \frac{6 \pm \sqrt{36-4}}{2} \qquad \lambda = -\frac{b \pm \sqrt{b^2-4ac}}{2a}$$

$$\lambda = 3 \pm \frac{\sqrt{32}}{2} = 3 \pm \frac{4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}$$

You Try It!

Q: What are the eigenvalues of A?

$$A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$$

Step 1: compute determinant of $A - \lambda I$

$p(\lambda) = \lambda^2 - (\text{trace}(A))\lambda + (\det A) = \lambda^2 - (2+(-1))\lambda + (-2-4) = \lambda^2 - \lambda - 6$

$$\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & -4 \\ -1 & -1-\lambda \end{pmatrix} = (2-\lambda)(-1-\lambda) - 4$$

$$= \lambda^2 - 2\lambda + \lambda - 2 - 4 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0$$

roots of $p(\lambda) = \det(A-\lambda I)$ are $\lambda_1 = 3$ and $\lambda_2 = -2$

↑ eigenvalues of A.

trick. $p(\lambda) =$

Problem: Find the characteristic polynomial of A and then find the eigenvalues of the matrix where $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$

$\lambda_1 = 3 \quad \lambda_2 = 2 \quad \lambda_3 = 1$

Step 1: compute $\det(A - \lambda I)$

$$\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 2 & 1 & 1-\lambda \end{pmatrix}$$

$$= (3-\lambda)(2-\lambda)(1-\lambda)$$

$$= (-1)^3(\lambda-3)(\lambda-2)(\lambda-1) = 0$$

$\Rightarrow \lambda = 3, 2, 1$ eigenvalues of A.

Recall from before

If A is upper or lower triangular, then the determinant of A is the product of the diagonal entries.

Consequently, if A is upper or lower triangular, then the eigenvalues of A are just the diagonal entries.

Warning! This trick does **not work** to find the eigenvalues unless A is upper/lower triangular!

The Characteristic Polynomial

Example

Question: What is the characteristic polynomial of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$?

Definition

The **trace** of a square matrix A is $\mathrm{Tr}(A)$ = sum of the diagonal entries of A.

Shortcut

The characteristic polynomial of a 2×2 matrix A is $f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A)$.

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \lambda^2 - (a+d)\lambda + (ad-bc)$$

5×5 $\quad p(\lambda) = -\lambda^5 + \mathrm{trace}A\,\lambda^4 - \cdots \pm \det A$

mystery!

Summary

We did two different things today.

First we talked about the geometry of eigenvalues and eigenvectors:

- Eigenvectors are vectors v such that v and Av are on the same line through the origin.
- You can pick out the eigenvectors geometrically if you have a picture of the associated transformation.

Then we talked about characteristic polynomials:

- We learned to find the eigenvalues of a matrix by computing the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.
- For a 2×2 matrix A, the characteristic polynomial is just

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A).$$