Section 5.4

Diagonalization

Motivation
Difference equations

Many real-word linear algebra problems have the form:

$v_1 = Av_0, \quad v_2 = Av_1 = A^2v_0, \quad v_3 = Av_2 = A^3v_0, \quad \ldots \quad v_k = Av_{k-1} = A^kv_0.$

This is called a **difference equation**.

Our toy example about rabbit populations had this form.

The question is, what happens to v_k as $n \to \infty$?

- Taking powers of diagonal matrices is easy!
- Taking powers of *diagonalizable* matrices is still easy!
- Diagonalizing a matrix is an eigenvalue problem.

Powers of Diagonal Matrices

If D is diagonal, then D^n is also diagonal; its diagonal entries are the nth powers of the diagonal entries of D:

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad D^2 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad D^3 = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}, \quad \ldots \quad D^n = \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix}.$$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad D^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix}, \quad D^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{27} \end{pmatrix},$$
$$\ldots \quad D^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & \frac{1}{3^n} \end{pmatrix}$$

Similar Matrices

Definition
Two $n \times n$ matrices are **similar** if there exists an invertible $n \times n$ matrix C such that $A = CBC^{-1}$.

A is similar to diagonal D if can find new invertible matrix C

Such that

$$A = CDC^{-1}$$

invertible · diagonal.

for n=3 and n=6

Powers of Matrices that are Similar to Diagonal Ones

What if A is not diagonal?

Example
Let $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$. Compute A^3, using

$$A = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

(continued on next page)

Diagonalization or why you should care about eigenvalues, anyway!

Exam 3
Next week
on Wed 11/18
@ 6:30 pm
in common exam rooms

$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

$D^2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 \\ 0 & (-1)^2 \end{bmatrix}$

$D^3 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}$

$D^2 * D$

$D^n = \begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix}$

Fact: if two matrices are similar then so are their powers:
$A = CBC^{-1} \implies A^n = CB^nC^{-1}$.

$A^2 = \overbrace{CDC^{-1}} \cdot \overbrace{CDC^{-1}}$

$= CDIDC^{-1}$

$= CD * D * C^{-1}$

$= CD^2C^{-1}$

$A^3 = \begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix}^3 = \begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix}$ ← harder? (for large n!)

$= \left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\right) =$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\cancel{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}}\cancel{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}\cancel{\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}}\cancel{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}}\cancel{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^3\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$ (easier?) for larger n!

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

C D C^{-1}

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{-1-1} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -8 & -8 \\ 1 & -1 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} -7 & -9 \\ -9 & -7 \end{bmatrix} = \boxed{\begin{bmatrix} 7/2 & 9/2 \\ 9/2 & 7/2 \end{bmatrix}} = A^3$$

for n=3 and n=6

Powers of Matrices that are Similar to Diagonal Ones

What if A is not diagonal?

Example

Let $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$. Compute A^6, using

$A = CDC^{-1}$ for $C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.

Long "naive" way

$A^6 = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$

$= A^3 A^3$

$A^6 = \begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix}^6$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^6\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 64 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

$= -\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -64 & -64 \\ -1 & 1 \end{bmatrix}$

$= -\frac{1}{2}\begin{bmatrix} -65 & -63 \\ -63 & -65 \end{bmatrix} = \begin{bmatrix} 65/2 & 63/2 \\ 63/2 & 65/2 \end{bmatrix} = A^6$

Punchline $C = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$ ← columns are eigenvectors of A

$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ ← diagonal entries are eigenvalues

Sanity check:

$Av_1 = \begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 + 3/2 \\ 3/2 + 1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓ $\lambda_1 = 2$ ✓

$Av_2 = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{bmatrix} 1/2 - 3/2 \\ 3/2 - 1/2 \end{bmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = -1$

Does it work?

$\begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$

New C New D New C^{-1}

Diagonalizable Matrices

Definition

An $n \times n$ matrix A is **diagonalizable** if it is similar to a diagonal matrix:

$A = CDC^{-1}$ for D diagonal.

Diagonalization

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors in $\mathbf{R}^n$.

In this case, $A = CDC^{-1}$ for

$$C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues (in the same order).

Diagonalization

Procedure

How to diagonalize a matrix A:

1. Find the eigenvalues of A using the characteristic polynomial.
2. For each eigenvalue λ of A, compute a basis $\mathcal{B}_\lambda$ for the λ-eigenspace.
3. If there are fewer than n total vectors in the union of all of the eigenspace bases $\mathcal{B}_\lambda$, then the matrix is not diagonalizable.
4. Otherwise, the n vectors $v_1, v_2, \ldots, v_n$ in your eigenspace bases are linearly independent, and $A = CDC^{-1}$ for

$$C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where λ_i is the eigenvalue for v_i.

Important

If $A = CDC^{-1}$ for $D = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$ then

$$A^k = CD^kC^{-1} = C\begin{pmatrix} d_{11}^k & 0 & 0 \\ 0 & d_{22}^k & 0 \\ 0 & 0 & d_{33}^k \end{pmatrix}C^{-1}.$$

$(v_i \neq 0 \quad Av_i = \lambda_i v_i)$

FACT: If $\lambda_1, \lambda_2, \lambda_3$ distinct eigenvalues then $\{v_1, v_2, v_3\}$ are linearly ind.

Corollary ← a theorem that follows easily from another theorem

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Warning! Knowing A has **repeated** eigenvalues does **NOT** mean that it is **not** diagonalizable!

Diagonalization
Example

want eigenvectors & eigenvalues!

Problem: Diagonalize $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$

The characteristic polynomial is

$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2).$ $\quad \lambda_1 = 2$ and $\lambda_2 = -1$

But how do you find C & D so that $A = CDC^{-1}$ in the first place?

Key IDEA: $A = CDC^{-1}$, C invertible, cols are eigenvectors of A

& D diagonal w/ λs on diagonal entries.

A: find eigenvalues & eigenvectors.

Start w/ char poly.

$$p(\lambda) = \det(A - \lambda I) = \lambda^2 - \mathrm{trace}(A)\lambda + \det A$$
$$= \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2) = 0$$
$$\Rightarrow \lambda_1 = 2 \quad \lambda_2 = -1$$

$\lambda_1 = 2$ $\quad A - \lambda I = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -3/2 & 3/2 \\ 3/2 & -3/2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ ✓ x_2

$X = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a basis for the $\lambda_1 = 2$ eigenspace

$\lambda_2 = -1$ $\quad A - \lambda I = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \end{bmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ ✓ x_2

$X = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

So basis for $\lambda_2 = -1$ eigenspace is

$v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$A = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

alternate diag'zation $= \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$

alt. diag'zation scale v_1, v_2. $= \begin{pmatrix} 2 & -3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 2 & 3 \end{pmatrix}^{-1}$

Diagonalization
Another example

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

The characteristic polynomial is

$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda-1)^2(\lambda-2).$ $= -(\lambda-1)(\lambda-1)(\lambda-2) = 0$

$\lambda_1 = 1$ $\quad \lambda_2 = 2$

multiplicity of 2

~~$\lambda_1 = 1$? $\lambda_2 = 1$? $\lambda_3 = 2$~~

Algebraic Multiplicity

Definition
The **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Diagonalization
Another example

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

The characteristic polynomial is

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda - 1)^2(\lambda - 2).$$

$\lambda_1 = 1$ $d_2 = 2$ eigenvalues

$\lambda_1 = 1$ $A - \lambda I = \begin{bmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & -3 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (x_2 free, x_3 free)

$x_1 - x_2 + x_3 = 0$

$x_2 = x_2$ free

$x_3 = x_3$ free

$\Rightarrow X = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

So basis for $\lambda_1 = 1$ eigenspace is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

And the eigenspace is a plane.

$\lambda_2 = 2$ $A - 2I = \begin{bmatrix} 2 & -3 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & +1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ ($-2R_3 + R_2$)

$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ (x_3 free)

$X = x_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

So $v_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector for A w/ $d_2 = 2$.

Check: $\begin{bmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 - 6 \\ 6 - 2 \\ 3 - 2 + 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ ✓

Algebraic Multiplicity

Definition
The **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

META: ① Find λ's by solving $p(\lambda) = \det(A - \lambda I) = 0$ to get eigenvalues

② for each eigenvalue row reduce $A - \lambda I$ & get parametric vector form of solns to $(A - \lambda I)x = 0$ to get a basis of eigenvectors.

③ form the matrices $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ & $C = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$

(& make sure to match order of columns of C w/ entries of D)

$$\begin{bmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}^{-1}}_{C^{-1}}$$

eigenvectors v_1, v_2 w/ $\lambda_1 = 1$ eigenvalue

Diagonalization

Procedure

The Diagonalization Theorem (Alternate Form)

Let A be an $n \times n$ matrix. The following are equivalent:

1. A is diagonalizable.
2. The sum of the geometric multiplicities of the eigenvalues of A equals n.
3. The sum of the algebraic multiplicities of the eigenvalues of A equals n, and for each eigenvalue, *the geometric multiplicity equals the algebraic multiplicity*.

Algebraic Multiplicity

Definition

The **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Non-Distinct Eigenvalues

Definition

Let λ be an eigenvalue of a square matrix A. The **geometric multiplicity** of λ is the dimension of the λ-eigenspace.

Theorem

Let λ be an eigenvalue of a square matrix A. Then

$1 \leq$ (the geometric multiplicity of λ) $\leq$ (the algebraic multiplicity of λ).

Non-Distinct Eigenvalues

Examples

Example

If A has n distinct eigenvalues, then the algebraic multiplicity of each equals 1, hence so does the geometric multiplicity, and therefore A is diagonalizable.

For example, $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$ has eigenvalues -1 and 2, so it is diagonalizable.

Example

The matrix $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ has characteristic polynomial

$$f(\lambda) = -(\lambda - 1)^3(\lambda - 2).$$

The algebraic multiplicities of 1 and 2 are 2 and 1, respectively. They sum to 3.

We showed before that the geometric multiplicity of 1 is 2 (the 1-eigenspace has dimension 2). The eigenvalue 2 automatically has geometric multiplicity 1.

Hence the geometric multiplicities add up to 3, so A is diagonalizable.

Diagonalization

A non-diagonalizable matrix

geo ≠ alg (for some λ)

Problem: Show that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Step 1: eigenvalues are $\lambda_1 = 1$ $\lambda_2 = 1$

$p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 0$

$= 0 \Rightarrow \lambda = 1$ alg mult of 2.

Step 2: $\lambda_1 = 1$ find eigenvector

$A - \lambda I = A - I = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (x_1 over the 1st column)

$x_1 = x_1$ (free)

$x_2 = 0$

$X = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (infinity many eigenvectors on this line span $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$)

geo mult. $\lambda = 1$ is 1

So A not diag'ble.

$$A = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

You Try It!

Poll

Poll

Which of the following matrices are diagonalizable, and why?

A. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ C. $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = A$

C. No same idea as A.

A. $\lambda = 1$ alg 2.

$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ geo mult. 1 (one free var of $A - \lambda I$)

geo ≠ alg and A NOT diag'ble.

B. Yes. $\lambda_1 = 1, \lambda_2 = 2$ alg mult of each λ_i is 1

geo mult. of each λ_i is 1.

D. ?? alg = geo $\lambda = 2$.

$A = I A I^{-1}$

inv! ↑ diag

More examples! Skip.

Problem: Diagonalize the matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Problem: Diagonalize the matrix $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$ which has characteristic polynomial $p(\lambda) = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2$

$\lambda_1 = 1$ alg 1 $\quad \lambda_2 = -2$ alg 2

$$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$$

$\lambda_1 = 1$ (alg 1): $A - I = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (x_3)

$X = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\lambda_2 = -2$ (alg 2): $A - (-2)I = A + 2I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (x_2, x_3)

$X = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

So $A \stackrel{?}{=} \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$??

$d_2 = -2$, $\lambda_1 = 1$

Match cols of C w/ entries of D.!!

You Try It!

Poll

Poll

Which of the following matrices are diagonalizable, and why?

A. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ C. $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Summary

- A matrix A is **diagonalizable** if it is similar to a diagonal matrix D: $A = CDC^{-1}$.
- It is easy to take powers of diagonalizable matrices: $A^t = CD^tC^{-1}$.
- An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$ in $\mathbf{R}^n$, in which case $A = CDC^{-1}$ for

$$C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

- If A has n distinct real eigenvalues, then it is diagonalizable.
- The **geometric multiplicity** of an eigenvalue λ is the dimension of the λ-eigenspace.
- $1 \leq$ (geometric multiplicity) $\leq$ (algebraic multiplicity).
- An $n \times n$ matrix is diagonalizable if and only if the sum of the geometric multiplicities is n.

EXAM 3 NEXT WEEK

On Wednesday, Nov 18 @ 6:30pm in the common exam room locations

Section 5.5

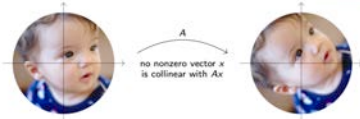
Complex Eigenvalues

A Matrix with No Eigenvectors

Consider the matrix for the linear transformation for rotation by $\pi/4$ in the plane. The matrix is:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = 45°$$

This matrix has no eigenvectors, as you can see geometrically: [interactive]



$$\det\begin{bmatrix} 1/\sqrt{2} - \lambda & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} - \lambda \end{bmatrix} = \left(\frac{1}{\sqrt{2}} - \lambda\right)^2 + \frac{1}{2}$$

$$= \lambda^2 - \sqrt{2}\lambda + \frac{1}{2} + \frac{1}{2}$$

$$= \lambda^2 - \sqrt{2}\lambda + 1$$

$$\lambda = \frac{\sqrt{2} \pm \sqrt{2-4}}{2}$$

$$= \frac{\sqrt{2} \pm \sqrt{-2}}{2}$$

$\sqrt{-2}$ not a real number!

Question: what are the eigenvalues of A?

$a+bi$ complex number. a, b real $\quad i = \sqrt{-1}$

Operations on Complex Numbers

Definition
The number i is defined such that $i^2 = -1$.

imaginary parts.

Addition: $(2-3i)+(-1+i) = 1 - 3i + i = 1 - 2i$

real parts

Multiplication: $(2-3i)(-1+i) = -2 + 3i + 2i - 3i^2$

$= -2 + 5i - 3(-1) = 1 + 5i$

Complex conjugation: $\overline{a+bi} = a - bi$ is the **complex conjugate** of $a + bi$.

Example: $\overline{2-3i} = 2 + 3i$

$\overline{-1+i} = -1 - i$

z_1, z_2 both complex

$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

$\overline{(2-3i)(-1+i)} \stackrel{?}{=} (2-3i)(-1+i)$

$1 + 5i \stackrel{?}{=} (2+3i)(-1-i)$

$1 - 5i = -2 - 2i - 3i - 3i^2 = -2 - 5i + 3 = 1 - 5i$ ✓

Division using conjugates

Example:

$$\frac{1+i}{1-i} = \frac{1+i}{1-i} * \frac{1+i}{1+i} = \frac{(1+i)^2}{1^2 - i^2} = \frac{1}{2}(1+i)^2 = \frac{1}{2}(1 + 2i + i^2)$$

$$= \frac{1}{2}(2i) = i$$

conjugates

$z * \bar{z}$ is REAL

$(a+bi)(a-bi) = a^2 - b^2 i^2 = a^2 + b^2$

The Fundamental Theorem of Algebra

The whole point of using complex numbers is to solve polynomial equations. It turns out that they are enough to find all solutions of all polynomial equations:

Fundamental Theorem of Algebra

Every polynomial of degree n has exactly n complex roots, counted with multiplicity.

Equivalently, if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then

$$f(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$$

for (not necessarily distinct) complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Important

If f is a polynomial with *real* coefficients, and if λ is a complex root of f, then so is $\bar{\lambda}$:

$$0 = \overline{f(\lambda)} = \overline{\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0}$$
$$= \bar{\lambda}^n + a_{n-1}\bar{\lambda}^{n-1} + \cdots + a_1\bar{\lambda} + a_0 = f(\bar{\lambda}).$$

Therefore complex roots of real polynomials come in *conjugate pairs*.

Consequently,

If λ is an eigenvalue of a real matrix A then so is $\bar{\lambda}$ the conjugate.

META: ① compute $A - \lambda I$ & row reduce to RREF
② get parametric vector form for solns to $(A - \lambda I)\vec{x} = 0$

Example: find an eigenvector for each eigenvalue.

$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ has eigenvalues $\lambda = \frac{1 \pm i}{\sqrt{2}}$.

$= \frac{\sqrt{2} \pm \sqrt{-2}}{2} = \frac{\sqrt{2}}{2} \pm \frac{\sqrt{-2}}{2} = \frac{\sqrt{2}}{2} \pm \frac{\sqrt{-1}\sqrt{2}}{2}$

$= \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2} i$

$\lambda_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

$\lambda_1 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$

Complex eigenvalues ALWAYS come in conjugate pairs!

λ_1 complex eigenvalue of A. Then so is $\bar{\lambda}_1$.

Start w/ λ_2

$\lambda_2 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$

$A - \lambda I = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} - \begin{pmatrix} \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i & 0 \\ 0 & \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i \end{pmatrix}$

$= \begin{bmatrix} \frac{\sqrt{2}}{2} i & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} i \end{bmatrix} \sim \begin{bmatrix} \sqrt{2} i & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} i \end{bmatrix} \sim \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$

$\sim \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \sim -i R_1 + R_2 \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$ (x_2 over the second column)

$-i(i) + (-1)$

$= -i^2 - 1 = -(-1) - 1$

$= 1 - 1 = 0$

parametric eqn form

$x_1 + i x_2 = 0$

$x_2 = x_2$ (free)

$X = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$

eigenvector $\lambda_2 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$, $\vec{v} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

for $\lambda_2 = \lambda_1$ eigenvector is $\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

Fact

Let A be a real square matrix. If λ is a complex eigenvalue with eigenvector v, then $\bar{\lambda}$ is an eigenvalue with eigenvector $\bar{v}$.

Why?

$$Av = \lambda \implies \overline{Av} = \overline{\lambda v} = \bar{\lambda}\bar{v}.$$

Both eigenvalues and eigenvectors of real square matrices occur in conjugate pairs.

META: ① Set $\det(A-\lambda I)=0$ & solve for λ to find eigenvalues

② Row reduce $A-\lambda I$ to RREF for λ_1

③ get parametric vector form for solns to $(A-\lambda I)\vec{x}=0$

④ use tricks $\vec{d}_1=\lambda_2$, $\vec{v}_1=\vec{v}_2$.

Problem: Find the eigenvalues of A and an eigenvector for each eigenvalue where $A=\begin{pmatrix}1 & -2\\ 5 & -5\end{pmatrix}$

$-5+10$

$p(\lambda)=\lambda^2+4\lambda+5=0$ $\quad \lambda=\frac{-4\pm\sqrt{16-20}}{2}$

$=-2\pm\frac{\sqrt{-4}}{2}$

$\lambda_2=-2-i$ $\quad =-2\pm\frac{2i}{2}=$ $\boxed{-2\pm i}$ λ's

$A-\lambda I=\begin{bmatrix}1 & -2\\ 5 & -5\end{bmatrix}-\begin{bmatrix}-2-i & 0\\ 0 & -2-i\end{bmatrix}$

$=\begin{bmatrix}1+2+i & -2\\ 5 & -5+2+i\end{bmatrix}=\begin{bmatrix}3+i & -2\\ 5 & -3+i\end{bmatrix}\sim\begin{bmatrix}5 & -3+i\\ 3+i & -2\end{bmatrix}$

$\sim\begin{bmatrix}1 & \frac{-3+i}{5}\\ 3+i & -2\end{bmatrix}$

$\sim -(3+i)R_1+R_2\begin{bmatrix}1 & \frac{-3+i}{5}\\ 0 & 0\end{bmatrix}$ (x_2)

$x=x_2\begin{pmatrix}(-3+i)/5\\ 1\end{pmatrix}=x_2\begin{pmatrix}\frac{3-i}{5}\\ 1\end{pmatrix}$.

You Try It!

Problem: Find the eigenvalues of A and an eigenvector for each eigenvalue where $A=\begin{pmatrix}3 & -5\\ 2 & -3\end{pmatrix}$

ans.

$\lambda_1=-i$ $\quad v_1=\begin{pmatrix}\frac{3-i}{2}\\ 1\end{pmatrix}$

$\lambda_2=i$ $\quad v_2=\begin{pmatrix}\frac{3+i}{2}\\ 1\end{pmatrix}$

λ_2 $\quad \vec{v}_2=\begin{pmatrix}\frac{3-i}{5}\\ 1\end{pmatrix}$

λ_1 $\quad \vec{v}_1=\begin{pmatrix}\frac{3+i}{5}\\ 1\end{pmatrix}$

True or False? If A is a 2×2 matrix with eigenvalue $\lambda_1 = 5$, then A cannot have eigenvalue $\lambda_2 = 2 + 3i$.

real

other eigenvalue must be real.

True or False? If A is a 2×2 matrix with eigenvalue $\lambda_1 = 5 + 3i$, then A cannot have eigenvalue $\lambda_2 = 2 + 3i$.

real

$\lambda_1 = \lambda_2$

eigenvalue of real 2×2 matrix A then $\lambda_2 = 2 - 3i$ has to be the other eigenvalue.

Summary

- One can do arithmetic with complex numbers just like real numbers: add, subtract, multiply, divide.
- An $n \times n$ matrix always exactly has complex n eigenvalues, counted with (algebraic) multiplicity.
- The complex eigenvalues and eigenvectors of a *real* matrix come in complex conjugate pairs:

$$A\vec{v} = \lambda\vec{v} \quad \Longrightarrow \quad A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}.$$