Section 5.6

Stochastic Matrices and PageRank

Reminder: Exam 3 on Wednesday 11/18 @ 6:30 pm in Common exam rooms

?? sample exams?

not on Exam 3 but YES on Final exam.

You'll be responsible for knowing basic facts about stochastic matrices, the Perron–Frobenius theorem, and PageRank, but we will not cover them in depth.

$$A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$$

30% probability a movie rented from location 3 gets returned to location 2

These are ALL stochastic matrices:

$\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ ✓ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓ $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ ✓

columns have non-negative entries that add to 1 / probability vectors as columns.

Stochastic Matrices

Example

Red Box has kiosks all over where you can rent movies. You can return them to any other kiosk. Let A be the matrix whose ij entry is the probability that a customer renting a movie from location j returns it to location i. For example,

the matrix A says

renting from Box 1 results in
- 30% returning to Box 1,
- 30% return to Box 2, and
- 40% return to Box 3

renting from Box 2 results in
- 40% return to Box 1,
- 40% returning to Box 2, and
- 20% return to Box 3

renting from Box 3 results in
- 50% return to Box 1,
- 30% return to Box 2,
- 20% returning to Box 3.

columns if where you START.

$$A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$$

row is where you END UP.

Try with Ae1, Ae2, Ae3 first!

100% I start @ Box 1.

$$\begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .3 \\ .3 \\ .4 \end{bmatrix}$$

Prob. of which box you end up with.

$$\begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .4 \\ .4 \\ .2 \end{bmatrix}$$

Start in Box 2 then where did I end up.

$$\begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}$$

Start in Box 3 Then 50% end up in Box 1.

$$Av = A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} .3x + .4y + .5z \\ .3x + .4y + .3z \\ .4x + .2y + .2z \end{pmatrix}$$

"The number of movies returned to location 2 will be (on average):
- 30% of the movies from location 1;
- 40% of the movies from location 2;
- 30% of the movies from location 3"

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} .3 \\ .3 \\ .4 \end{bmatrix} + y\begin{bmatrix} .4 \\ .4 \\ .2 \end{bmatrix} + z\begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}$$

$$A\begin{pmatrix} .50 \\ .50 \\ 0 \end{pmatrix} = .50\begin{bmatrix} .3 \\ .3 \\ .4 \end{bmatrix} + .50\begin{bmatrix} .4 \\ .4 \\ .2 \end{bmatrix} = \begin{bmatrix} .35 \\ .35 \\ .3 \end{bmatrix}$$

Stochastic Matrices and Difference Equations

If x_n, y_n, z_n are the numbers of movies in locations 1, 2, 3, respectively, on day n, and $v_n = (x_n, y_n, z_n)$, then:

$$v_n = Av_{n-1} = A^2v_{n-2} = \cdots = A^nv_0.$$

Recall: This is an example of a **difference equation**.

In general: A difference equation $v_{n+1} = Av_n$ is used to model a state change controlled by a matrix:
- v_0 is the "state at time n",
- v_{n+1} is the "state at time $n+1$", and
- $v_{n+1} = Av_n$ means that A is the "change of state matrix."

Matrix Expression and State Vector

$v_0 = \begin{pmatrix} 50 \\ 50 \\ 0 \end{pmatrix}$

$Av_0 = \begin{pmatrix} 35 \\ 35 \\ 30 \end{pmatrix}$

$A^2v_0 = \begin{pmatrix} 39.5 \\ 33.5 \\ 27 \end{pmatrix}$

$A^3v_0 = \begin{pmatrix} 38.75 \\ 33.35 \\ 27.9 \end{pmatrix}$

$A^{10}v_0 \approx \begin{pmatrix} 38.88889 \\ 33.33333 \\ 27.77778 \end{pmatrix}$

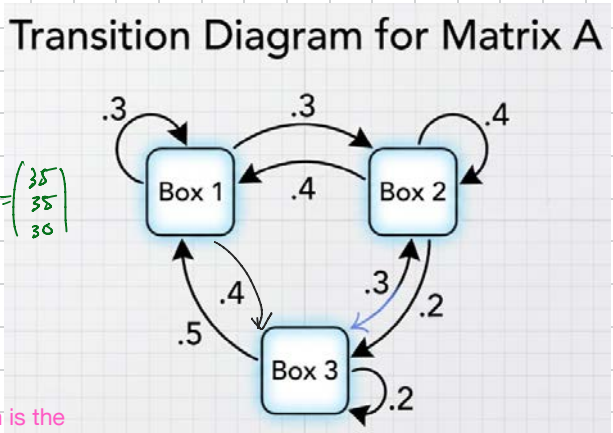
$A^{20}v_0 \approx \begin{pmatrix} 38.88889 \\ 33.33333 \\ 27.77778 \end{pmatrix}$ ← long term trend

$A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$

$v_1 = Av_0$

$= \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix} \begin{pmatrix} 50 \\ 50 \\ 0 \end{pmatrix} = \begin{pmatrix} 15+20 \\ 15+20 \\ 20+10 \end{pmatrix} = \begin{pmatrix} 35 \\ 35 \\ 30 \end{pmatrix}$

$v_2 = Av_1 = A^2v_0 = \begin{bmatrix} 39.5 \\ 33.5 \\ 27 \end{bmatrix}$



Note: the sum of the entries is always N=100 which is the total number of movies in circulation for this problem.

$$\begin{pmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{pmatrix} \cdot \begin{pmatrix} 50 \\ 50 \\ 0 \end{pmatrix} = \begin{pmatrix} 35 \\ 35 \\ 30 \end{pmatrix}$$

Q: Find the long term trend for the Red Box problem.

Key observation: need to find an eigenvector for $\lambda = 1$.

Diagonalizable Stochastic Matrices

The Red Box matrix $A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$ has characteristic polynomial

$$f(\lambda) = -\lambda^3 + 0.12\lambda - 0.02 = -(\lambda - 1)(\lambda + 0.2)(\lambda - 0.1).$$

Find λ's

$\lambda_1 = 1, \lambda_2 = -0.2, \lambda_3 = 0.1$

($|\lambda| \leq 1$ for stochastic matrix A & at least one $\lambda = 1$ eigenvalue)

META: Step 1 - row reduce A-I and find an eigenvector for lambda=1

$$A - I = \begin{pmatrix} -0.7 & 0.4 & 0.5 \\ 0.3 & -0.6 & 0.3 \\ 0.4 & 0.2 & -0.8 \end{pmatrix} \sim \begin{pmatrix} -7 & 4 & 5 \\ 3 & -6 & 3 \\ 4 & 2 & -8 \end{pmatrix} \sim \begin{pmatrix} -7 & 4 & 5 \\ 1 & -2 & 1 \\ 4 & 2 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ -7 & 4 & 5 \\ 4 & 2 & -8 \end{pmatrix}$$

x_3

$$\sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & -10 & 12 \\ 4 & 2 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & -10 & 12 \\ 0 & 10 & -12 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & -10 & 12 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1.4 \\ 0 & 1 & -1.2 \\ 0 & 0 & 0 \end{pmatrix}$$

META: Step 2 - scale your eigenvector so that the sum of the entries equals the total number of objects in the problem (in this case there were N=100 movies in circulation)

$x = x_3 \begin{bmatrix} 1.4 \\ 1.2 \\ 1 \end{bmatrix}$

Scale $\begin{bmatrix} 1.4 \\ 1.2 \\ 1 \end{bmatrix}$ to be a probability vector

Big IDEA!!

w_1, w_2, w_3 eigenvectors for $\lambda_1, \lambda_2, \lambda_3$

$v_0 = c_1w_1 + c_2w_2 + c_3w_3$

$\Rightarrow Av_0 = c_1Aw_1 + c_2Aw_2 + c_3Aw_3$

$= c_1\lambda_1w_1 + c_2\lambda_2w_2 + c_3\lambda_3w_3$

$\Rightarrow A^2v_0 = c_1\lambda_1^2w_1 + c_2\lambda_2^2w_2 + c_3\lambda_3^2w_3$

$x_3 = \frac{1}{3.6}$ $p = \frac{1}{3.6}\begin{bmatrix} 1.4 \\ 1.2 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/36 \\ 12/36 \\ 10/36 \end{bmatrix}$

$N = 100$, total # objects in each state

$100 \begin{pmatrix} 14/36 \\ 12/36 \\ 10/36 \end{pmatrix} \approx \begin{bmatrix} 38.89 \\ 33.33 \\ 27.78 \end{bmatrix}$

$A^{20}v_0 \approx \begin{pmatrix} 38.88889 \\ 33.33333 \\ 27.77778 \end{pmatrix}$

Diagonalizable Stochastic Matrices

Example from §5.3

Let $A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$ This is a positive stochastic matrix.

This matrix is diagonalizable:

$$A = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

$w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda_1 = 1$

$w_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\lambda_2 = \frac{1}{2}$

Question: Find the long term trend for the initial vector v0=[35 ; 15]

META: Step 1 - row reduce A-I and find an eigenvector for lambda=1
Step 2 - scale your eigenvector so that the sum of the entries equals the total number of objects in the problem (in this case there are N=50 objects in circulation)

for $V_0 = \begin{bmatrix} 35 \\ 15 \end{bmatrix}$ the long term trend $A^n V_0 \approx N \hat{w}_1$

where N = total number of objects &
$\hat{w}_1$ = scaled eigenvector w/ entries that add to 1.

$\hat{w}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 50\% \\ 50\% \end{bmatrix}$

need to scale by N = 50

$50\,\hat{w}_1 = \begin{bmatrix} 25 \\ 25 \end{bmatrix}$

Q: why does this work? Hint: write v0 as a linear combination of eigenvectors and repeatedly apply A.

$\begin{pmatrix} 35 \\ 15 \end{pmatrix} = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Leftrightarrow \left[\begin{array}{cc|c} 1 & 1 & 35 \\ 1 & -1 & 15 \end{array}\right]$ $C_1 = 25$, $C_2 = 10$

$\sim \left[\begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -20 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & 0 & 25 \\ 0 & 1 & 10 \end{array}\right]$

$\begin{bmatrix} 35 \\ 15 \end{bmatrix} = 25\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$A\begin{bmatrix} 35 \\ 15 \end{bmatrix} = 25\,A\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10\,A\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$= 25\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} = 25\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{10}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$A^2\begin{pmatrix} 35 \\ 15 \end{pmatrix} = 25\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{10}{2^2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\vdots$

$A^n\begin{pmatrix} 35 \\ 15 \end{pmatrix} = 25\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{10}{2^n}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ → 0

↑ wins

in the long run just get $A^n\begin{pmatrix} 35 \\ 15 \end{pmatrix} \approx \begin{pmatrix} 25 \\ 25 \end{pmatrix}$

Stochastic Matrices and Difference Equations

If x_n, y_n, z_n are the numbers of movies in locations 1, 2, 3, respectively, on day n, and $v_n = (x_n, y_n, z_n)$, then:

$$v_n = Av_{n-1} = A^2 v_{n-2} = \cdots = A^n v_0.$$

Recall: This is an example of a **difference equation**.

Red Box probably cares about what v_n is as n gets large: it tells them where the movies will end up *eventually*. This seems to involve computing A^n for large n, but as we will see, they actually only have to compute one eigenvector.

In general: A difference equation $v_{n+1} = Av_n$ is used to model a state change controlled by a matrix:

- v_n is the "state at time n",
- v_{n+1} is the "state at time $n+1$", and
- $v_{n+1} = Av_n$ means that A is the "change of state matrix."

Why is it enough to find an eigenvector for λ=1 if you want to find the long term trend?

Eigenvalues of Stochastic Matrices

Fact: 1 is an eigenvalue of a stochastic matrix.

Why? If A is stochastic, then 1 is an eigenvalue of A^T:

$$\begin{pmatrix} .3 & .3 & .4 \\ .4 & .4 & .2 \\ .5 & .3 & .2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Lemma
A and A^T have the same eigenvalues.

Proof: $\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I)$, so they have the same characteristic polynomial.

Eigenvalues of Stochastic Matrices
Continued

Fact: if λ is an eigenvalue of a stochastic matrix, then $|\lambda| \leq 1$. Hence 1 is the *largest* eigenvalue (in absolute value).

Diagonalizable Stochastic Matrices
Example from §5.3

Let $A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$. This is a positive stochastic matrix.

This matrix is diagonalizable:

$$A = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Let $w_1 = \binom{1}{1}$ and $w_2 = \binom{1}{-1}$ be the columns of C.

$$A\left(a_1\begin{pmatrix}1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\-1\end{pmatrix}\right) = a_1\begin{pmatrix}1\\1\end{pmatrix} + \tfrac{1}{2}a_2\begin{pmatrix}1\\-1\end{pmatrix}$$
$$A^2\left(a_1\begin{pmatrix}1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\-1\end{pmatrix}\right) = a_1\begin{pmatrix}1\\1\end{pmatrix} + \tfrac{1}{4}a_2\begin{pmatrix}1\\-1\end{pmatrix}$$
$$A^3\left(a_1\begin{pmatrix}1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\-1\end{pmatrix}\right) = a_1\begin{pmatrix}1\\1\end{pmatrix} + \tfrac{1}{8}a_2\begin{pmatrix}1\\-1\end{pmatrix}$$
$$\vdots$$
$$A^n\left(a_1\begin{pmatrix}1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\-1\end{pmatrix}\right) = a_1\begin{pmatrix}1\\1\end{pmatrix} + \tfrac{1}{2^n}a_2\begin{pmatrix}1\\-1\end{pmatrix}$$

When n is large, the second term disappears, so $A^n x$ approaches $a_1 w_1$, which is an *eigenvector with eigenvalue* 1 (assuming $a_1 \neq 0$). So all vectors get "sucked into the 1-eigenspace," which is spanned by $w_1 = \binom{1}{1}$.

$$A\left(a_1\begin{pmatrix}1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\-1\end{pmatrix}\right) = a_1\begin{pmatrix}1\\1\end{pmatrix} + \tfrac{1}{2}a_2\begin{pmatrix}1\\-1\end{pmatrix}$$
$$A^2\left(a_1\begin{pmatrix}1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\-1\end{pmatrix}\right) = a_1\begin{pmatrix}1\\1\end{pmatrix} + \tfrac{1}{4}a_2\begin{pmatrix}1\\-1\end{pmatrix}$$
$$A^3\left(a_1\begin{pmatrix}1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\-1\end{pmatrix}\right) = a_1\begin{pmatrix}1\\1\end{pmatrix} + \tfrac{1}{8}a_2\begin{pmatrix}1\\-1\end{pmatrix}$$
$$\vdots$$
$$A^n\left(a_1\begin{pmatrix}1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\-1\end{pmatrix}\right) = a_1\begin{pmatrix}1\\1\end{pmatrix} + \tfrac{1}{2^n}a_2\begin{pmatrix}1\\-1\end{pmatrix}$$

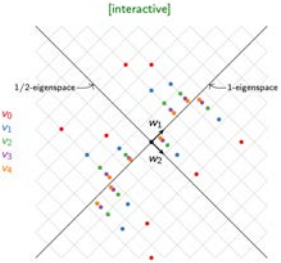
https://textbooks.math.gatech.edu/ila/demos/dynamics.html?mat=1,0;0.1/2&v1=1,1,0&v2=1,-1,0

Diagonalizable Stochastic Matrices
Example, continued

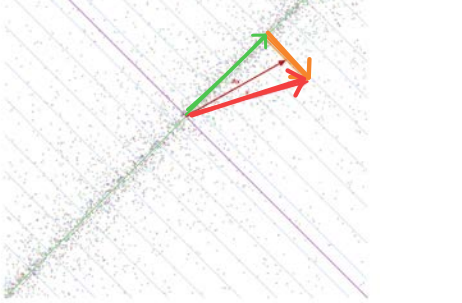




[interactive]



All vectors get "sucked into the 1-eigenspace."



Perron–Frobenius Theorem

Definition
A *steady state* for a stochastic matrix A is an eigenvector w with eigenvalue 1, such that all entries are *positive* and sum to 1.

Perron–Frobenius Theorem
If A is a positive stochastic matrix, then it admits a unique steady state vector w, which spans the 1-eigenspace.

Moreover, for any vector v_0 with entries summing to some number c, the iterates $v_1 = Av_0$, $v_2 = Av_1$, ..., $v_n = Av_{n-1}$, ..., approach cw as n gets large.

Translation: The Perron–Frobenius Theorem says the following:

- The 1-eigenspace of a positive stochastic matrix A is a line.
- To compute the steady state, find any 1-eigenvector (as usual), then divide by the sum of the entries; the resulting vector w has entries that sum to 1, and are *automatically* positive.
- Think of w as a vector of steady state *percentages*: if the movies are distributed according to these percentages today, then they'll be in the same distribution tomorrow.
- The sum c of the entries of v_0 is the total number of movies; eventually, the movies arrange themselves according to the steady state percentage, i.e., $v_n \to cw$.

a probability vector w/ eigenvalue $\lambda = 1$.

Let's spend the rest of the day solving sample exam problems!

Spring 2022

d) (2 points) Consider the positive stochastic matrix $A = \begin{pmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{pmatrix}$. The 1-eigenspace of A is spanned by $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. What vector does $A^n \begin{pmatrix} 17 \\ 4 \end{pmatrix}$ approach as n gets very large?

(i) $\begin{pmatrix} 9 \\ 12 \end{pmatrix}$ (ii) $\begin{pmatrix} 17 \\ 4 \end{pmatrix}$ (iii) $\begin{pmatrix} 51 \\ 16 \end{pmatrix}$ (iv) $\begin{pmatrix} 9/21 \\ 12/21 \end{pmatrix}$ (v) $\begin{pmatrix} 51/67 \\ 16/67 \end{pmatrix}$ (vi) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

b) Amara's coffee shop competes with Ben's cafe for customers. Currently in 2022, Amara has 210 customers and Ben has 140 customers.

Each year:

- 70% of Amara's customers stay with Amara, while 30% of Amara's customers switch to Ben.
- 60% of Ben's customers stay with Ben, while 40% of Ben's customers switch to Amara.

Answer the following questions.

(i) (4 points) Write a positive stochastic matrix A and a vector x so that Ax is the vector that gives the number of customers for Amara and Ben (in that order) in 2023. Do not carry out the multiplication! Just write A and x.

(ii) (3 points) Find the steady-state vector w for the matrix A.

(b) Step 1: write down A stochastic matrix.

Step 2: find a $\lambda = 1$ eigenvector by row reducing $A - I$.

Step 3: Scale the eigenvector to get entries that add to 1 or to N.

c) (3 points) Suppose A is a 2×2 positive stochastic matrix with the property that as n gets very large, $A^n \begin{pmatrix} 90 \\ 70 \end{pmatrix}$ approaches $\begin{pmatrix} 120 \\ 40 \end{pmatrix}$.

What is the steady-state vector w for A?

Problem 7.

Free response. Show your work! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

a) Awesome Coffee and Bunk Coffee compete for a market of 60 customers who drink coffee every day. Today, Awesome Coffee has 50 daily customers and Bunk Coffee has 10 daily customers.

Each day:

- 60% of Awesome Coffee customers keep drinking Awesome Coffee, while 40% switch to Bunk Coffee.
- 80% of Bunk Coffee customers keep drinking Bunk Coffee, while 20% switch to Awesome Coffee.

(i) (3 points) Write a positive stochastic matrix A **and** a vector x so that Ax will give the number of customers for Awesome Coffee and Bunk Coffee (in that order) tomorrow. You do not need to compute Ax.

$$A = \begin{pmatrix} & \\ & \end{pmatrix} \qquad x = \begin{pmatrix} \\ \end{pmatrix}$$

(ii) (4 points) In the long term, approximately how many daily customers will Awesome Coffee have?

To receive full credit, you needed to answer the question that was directly asked. This means that your answer must be a **number**, not a vector. If you just wrote $\begin{pmatrix} 20 \\ 40 \end{pmatrix}$ and did not interpret that vector to answer the question, you did not receive full credit.

b) (5 points) Axel and Billy are magicians who compete for customers in a group of 200 people. Today, Axel has 160 customers and Billy has 40 customers. Each day:

- 30% of Axel's customers keep attending Axel's show, while 70% of Axel's customers switch to Billy's show.
- 80% of Billy's customers attend Billy's show, while 20% of Billy's customers switch to Axel's show.

(i) Write a positive stochastic matrix B and a vector x so that Bx will give the number of customers for Axel's show and Billy's show (in that order) tomorrow. You do not need to compute Bx.

$$B = \begin{pmatrix} & \\ & \end{pmatrix} \qquad x = \begin{pmatrix} \\ \end{pmatrix}$$

(ii) Find the steady-state vector w for B. Write your answer in the space below.

$$w = \begin{pmatrix} \\ \end{pmatrix}$$

4. On this page, you do not need to show your work. Only your answers are graded. Parts (a)-(d) are unrelated.

   (a) (3 points) Suppose A is a 6×6 matrix and its characteristic polynomial is

   $$\det(A - \lambda I) = \lambda^2(5 - \lambda)^3(2 - \lambda).$$

   Which of the following statements are true? Fill in the bubble for all that apply.

   ◯ If the null space of A is a line, then A cannot be diagonalizable.

   ◯ A cannot be a stochastic matrix.

   ◯ The solution set to the equation $Ax = 2x$ is a line.

7. Free response. Show your work except on part (a) part (i)! A correct answer elsewhere without sufficient work will receive little or no credit. Parts (a) and (b) are unrelated.

   (a) (7 points) The Great Journey is an online game where participants play either as the Bard or the Cleric. A total of 110 people play the game. This year, 80 players use the Bard and 30 players use the Cleric. Each year:

   - 60% of players who use the Bard keep playing as the Bard, while 40% switch to the Cleric.
   - 30% of players who use the Cleric keep playing as the Cleric, while 70% switch to the Bard.

   (i) Write a positive stochastic matrix A and a vector x so that Ax will give the number of players for the Bard and the Cleric (in that order) next year. You do not need to compute Ax.

   $$A = \begin{pmatrix} & \\ & \end{pmatrix} \qquad x = \begin{pmatrix} \\ \end{pmatrix}$$

   (ii) In the long run, roughly how many yearly players will the Cleric have?

   Enter your answer here: _____.

Summary

- **Stochastic** and **positive stochastic** matrices model probabilistic systems.
- We care about the long-term behavior of such a system. This is called the **steady state**. It tells us the eventual state of the system.
- The Perron–Frobenius theorem says that a positive stochastic matrix always has a unique steady state.
- If you can understand the RedBox example, then you understand almost everything.
- The Google matrix is an example of a positive stochastic matrix.
- The steady state of the Google matrix is the PageRank.

Real world application - Google searches!

Google's PageRank

Internet searching in the 90's was a pain. Yahoo or AltaVista would scan pages for your search text, and just list the results with the most occurrences of those words.

Not surprisingly, the more unsavory websites soon learned that by putting the words "Alanis Morissette" a million times in their pages, they could show up first every time an angsty teenager tried to find *Jagged Little Pill* on Napster.

Larry Page and Sergey Brin invented a way to rank pages by *importance*. They founded Google based on their algorithm.

Here's how it works. (roughly)

Reference:

http://www.math.cornell.edu/~mec/Winter2009/RalucaRemus/Lecture3/lecture3.html

The Importance Rule

Each webpage has an associated importance, or **rank**. This is a positive number.

The Importance Rule

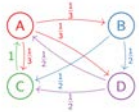
If page P links to n other pages $Q_1, Q_2, \ldots, Q_n$, then each Q_i should inherit $\frac{1}{n}$ of P's importance.

- So if a very important page links to your webpage, your webpage is considered important.
- And if a ton of unimportant pages link to your webpage, then it's still important.
- But if only one crappy site links to yours, your page isn't important.

Random surfer interpretation: a "random surfer" just sits at his computer all day, randomly clicking on links. The pages he spends the most time on should be the most important. This turns out to be equivalent to the rank.

The Importance Matrix

Consider the following Internet with only four pages. Links are indicated by arrows.



Page A has 3 links, so it passes $\frac{1}{3}$ of its importance to pages B, C, D.
Page B has 2 links, so it passes $\frac{1}{2}$ of its importance to pages C, D.
Page C has one link, so it passes all of its importance to page A.
Page D has 2 links, so it passes $\frac{1}{2}$ of its importance to pages A, C.

In terms of matrices, if $v = (a, b, c, d)$ is the vector containing the ranks a, b, c, d of the pages A, B, C, D, then

Importance Rule

importance matrix: ij entry is importance page j passes to page i

$$\begin{pmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} c + \frac{1}{2}d \\ \frac{1}{3}a \\ \frac{1}{3}a + \frac{1}{2}b + \frac{1}{2}d \\ \frac{1}{3}a + \frac{1}{2}b \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Adjustment #1: if any pages have no outward links, then a user picks any page at random (could pick the same page)

Adjustment #2: a user will go to a random page p=15% of the time and click a link 85% of the time.

p=15% is called the **damping factor**

Start with importance matrix A (no Adjustment#1 needed since every page has at least one outward link)

$$A = \begin{pmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$G = 0.85 \begin{pmatrix} 0 & 0 & 1 & 0.5 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0.5 & 0 & 0.5 \\ 1/3 & 0.5 & 0 & 0 \end{pmatrix} + (0.15) \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$

$$= 0.85 \begin{pmatrix} 0 & 0 & 1 & 0.5 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0.5 & 0 & 0.5 \\ 1/3 & 0.5 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0.0375 & 0.0375 & 0.0375 & 0.0375 \\ 0.0375 & 0.0375 & 0.0375 & 0.0375 \\ 0.0375 & 0.0375 & 0.0375 & 0.0375 \\ 0.0375 & 0.0375 & 0.0375 & 0.0375 \end{pmatrix}$$

$$G \approx \begin{pmatrix} 0.0375 & 0.0375 & 0.8875 & 0.4625 \\ 0.32083 & 0.0375 & 0.0375 & 0.0375 \\ 0.32083 & 0.4625 & 0.0375 & 0.4625 \\ 0.32083 & 0.4625 & 0.0375 & 0.0375 \end{pmatrix}$$

PageRank vector

$$\begin{pmatrix} 0.36815 \\ 0.14181 \\ 0.28796 \\ 0.20208 \end{pmatrix}$$



The Google Matrix

Upshot

Lemma
The Google matrix is a positive stochastic matrix.

The PageRank vector is the steady state for the Google Matrix.

This exists and has positive entries by the Perron–Frobenius theorem. The hard part is calculating it: the Google matrix has 1 gazillion rows.

Powers of the Google Matrix G^n

Matrix Expression and State Matrix

$$G^1 = \begin{pmatrix} 0.03750 & 0.03750 & 0.88750 & 0.46250 \\ 0.32083 & 0.03750 & 0.03750 & 0.03750 \\ 0.32083 & 0.46250 & 0.03750 & 0.46250 \\ 0.32083 & 0.46250 & 0.03750 & 0.03750 \end{pmatrix}$$

$$G^2 = \begin{pmatrix} 0.44656 & 0.62719 & 0.08531 & 0.44656 \\ 0.04812 & 0.04812 & 0.28896 & 0.16854 \\ 0.32083 & 0.26062 & 0.32083 & 0.20042 \\ 0.18448 & 0.06406 & 0.30490 & 0.18448 \end{pmatrix}$$

$$G^3 = \begin{pmatrix} 0.38861 & 0.28626 & 0.43979 & 0.28626 \\ 0.16403 & 0.21520 & 0.06167 & 0.16403 \\ 0.26288 & 0.26288 & 0.31406 & 0.31406 \\ 0.18448 & 0.23566 & 0.18448 & 0.23566 \end{pmatrix}$$

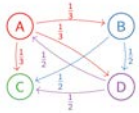
$$G^{10} \approx \begin{pmatrix} 0.36815 & 0.36866 & 0.36770 & 0.36844 \\ 0.14166 & 0.14147 & 0.14216 & 0.14182 \\ 0.28808 & 0.28800 & 0.28789 & 0.28783 \\ 0.20211 & 0.20186 & 0.20226 & 0.20192 \end{pmatrix}$$

$$G^{20} \approx \begin{pmatrix} 0.36815 & 0.36815 & 0.36815 & 0.36815 \\ 0.14181 & 0.14181 & 0.14181 & 0.14181 \\ 0.28796 & 0.28796 & 0.28796 & 0.28796 \\ 0.20208 & 0.20208 & 0.20208 & 0.20208 \end{pmatrix}$$

New example: remove the link out of website "C"

The Google Matrix
Example

Consider this Internet:



The importance and modified importance matrices are

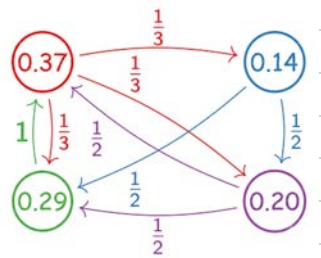
$$A = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{pmatrix} \xrightarrow{\text{modify}} A' = \begin{pmatrix} 0 & 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{4} & 0 \end{pmatrix}$$

If we choose the damping factor $p = .15$, then the Google matrix is

$$M = (1-p)A' + pB = \begin{pmatrix} 0.0375 & 0.0375 & 0.2500 & 0.4625 \\ 0.3208 & 0.0375 & 0.2500 & 0.0375 \\ 0.3208 & 0.4625 & 0.2500 & 0.4625 \\ 0.3208 & 0.4625 & 0.2500 & 0.0375 \end{pmatrix}$$

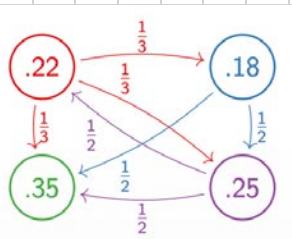
PageRank vector for G w/ link out of "C"

$$\begin{pmatrix} 0.36815 \\ 0.14181 \\ 0.28796 \\ 0.20208 \end{pmatrix}$$



PageRank vector for G with removed link out of "C"

$$\begin{pmatrix} 0.2192 \\ 0.1752 \\ 0.3558 \\ 0.2498 \end{pmatrix}$$

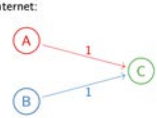


Why do we need a damping factor of p=15%?

Problems with the Importance Matrix
Dangling pages

Observation: the importance matrix is *not* positive: it's only nonnegative. So we can't apply the Perron–Frobenius theorem. Does this cause problems? Yes!

Consider the following Internet:



Adjustment #1: if any pages have no outward links, then a user picks any page at random (could pick the same page)

Replace any columns of zeros with 1/3's

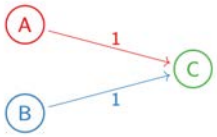
Adjustment #2: a user will go to a random page p=15% of the time and click a link 80% of the time.

Compute G=0.85*A+0.15*M where M has all 1/3's

You Try It!

Problem: Find the importance matrix A for the internet above with three websites A, B, C. Then, find the Google matrix for this internet (it is ok to leave your answer in the form $G = (0.85)A + (0.15)K$. Finally, think about the steps needed to find the steady-state vector for G in order to determine the PageRank vector.

SPOILERS! Answer to previous YouTryIt below.

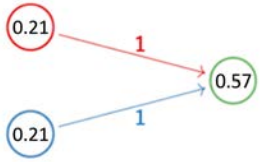


The importance matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

$$A_* = \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \\ 1 & 1 & \frac{1}{3} \end{pmatrix}$$

$$G = 0.85 \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 0 & 1/3 \\ 1 & 1 & 1/3 \end{pmatrix} + (0.15) \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$G = \begin{pmatrix} 0.05 & 0.05 & 0.3\overline{3} \\ 0.05 & 0.05 & 0.3\overline{3} \\ 0.90 & 0.90 & 0.3\overline{3} \end{pmatrix} \quad q \approx \begin{pmatrix} 0.2128 \\ 0.2128 \\ 0.5745 \end{pmatrix}$$



Why do we need a damping factor of p=15%?

Problems with the Importance Matrix
Disconnected internet

Consider the following Internet:



Adjustment #1: if any pages have no outward links, then a user picks any page at random (could pick the same page)

Replace any columns of zeros with 1/3's

Adjustment #2: a user will go to a random page p=15% of the time and click a link 80% of the time.

Compute G=0.85*A+0.15*M where M has all 1/3's

You Try It!

Problem: Find the importance matrix A for the internet above with five websites A, B, C, D, E. Then, find the Google matrix for this internet (it is ok to leave your answer in the form $G = (0.85)A + (0.15)K$. Finally, think about the steps needed to find the steady-state vector for G in order to determine the PageRank vector.

NOTE: Try to guess the PageRank vector just from the diagram!!!

$$A = A_* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

$$G = 0.85 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix} + (0.15) \begin{pmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{pmatrix}$$

$$G = \begin{pmatrix} 0.03 & 0.88 & 0.03 & 0.03 & 0.03 \\ 0.88 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.455 & 0.455 \\ 0.03 & 0.03 & 0.455 & 0.03 & 0.455 \\ 0.03 & 0.03 & 0.455 & 0.455 & 0.03 \end{pmatrix} \quad q = \begin{pmatrix} 0.20 \\ 0.20 \\ 0.20 \\ 0.20 \\ 0.20 \end{pmatrix}$$

Summary

- **Stochastic** and **positive stochastic** matrices model probabilistic systems.
- We care about the long-term behavior of such a system. This is called the **steady state**. It tells us the eventual state of the system.
- The Perron–Frobenius theorem says that a positive stochastic matrix always has a unique steady state.
- If you can understand the RedBox example, then you understand almost everything.
- The Google matrix is an example of a positive stochastic matrix.
- The steady state of the Google matrix is the PageRank.

Chapter 6

Orthogonality

Section 6.1

Dot Products and Orthogonality

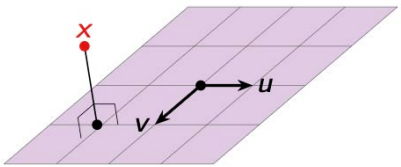
Orientation

Recall: This course is about learning to:

- Solve the matrix equation $Ax = b$
- Solve the matrix equation $Ax = \lambda x$
- Almost solve the equation $Ax = b$

We are now aiming at the last topic.

Idea: In the real world, data is imperfect. Suppose you measure a data point x which you know for theoretical reasons must lie on a plane spanned by two vectors u and v.



Goal: given x in R^n, find the closest vector in Span{u,v} to the vector x

The Dot Product

Definition

The **dot product** of two vectors x, y in $\mathbf{R}^n$ is

$$x \cdot y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \stackrel{\text{def}}{=} x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Thinking of x, y as column vectors, this is the same as $x^T y$.

Example

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} =$$

Example

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} =$$

Properties of the Dot Product

Many usual arithmetic rules hold, as long as you remember you can only dot two vectors together, and that the result is a scalar.

- $x \cdot y = y \cdot x$
- $(x + y) \cdot z = x \cdot z + y \cdot z$
- $(cx) \cdot y = c(x \cdot y)$

Dotting a vector with itself is special:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1^2 + x_2^2 + \cdots + x_n^2.$$

Hence:

- $x \cdot x \geq 0$
- $x \cdot x = 0$ if and only if $x = 0$.

Important: $x \cdot y = 0$ does *not* imply $x = 0$ or $y = 0$. For example, $\binom{1}{0} \cdot \binom{0}{1} = 0$.

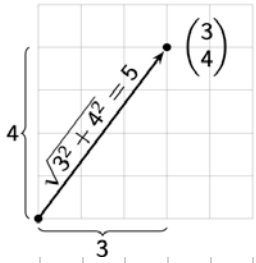
The Dot Product and Length

Definition
The **length** or **norm** of a vector x in $\mathbf{R}^n$ is

$$\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Fact
If x is a vector and c is a scalar, then $\|cx\| = |c| \cdot \|x\|$.

Example: find the length of 2*[3;4]

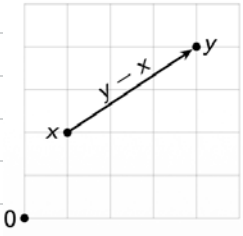


$$\left\| \begin{pmatrix}3\\4\end{pmatrix} \right\| =$$

The Dot Product and Distance

Definition
The **distance** between two points x, y in $\mathbf{R}^n$ is

$$\text{dist}(x, y) = \|y - x\|.$$



Example

Let $x = (1, 2)$ and $y = (4, 4)$. Then

$\text{dist}(x, y) =$

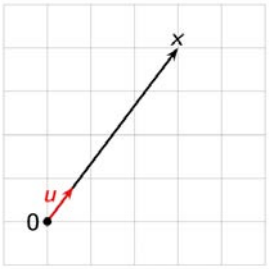
Unit Vectors

Definition
A **unit vector** is a vector v with length $\|v\| = 1$.

Example
The unit coordinate vectors are unit vectors:

$$\|e_1\| = \left\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\| = \sqrt{1^2 + 0^2 + 0^2} = 1$$

Definition
Let x be a nonzero vector in $\mathbf{R}^n$. The **unit vector in the direction of** x is the vector $\dfrac{x}{\|x\|}$.



Example

What is the unit vector in the direction of $x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$?

Orthogonality

Definition
Two vectors x, y are **orthogonal** or **perpendicular** if $x \cdot y = 0$.
Notation: $x \perp y$ means $x \cdot y = 0$.

Fact: $x \perp y \iff \|x - y\|^2 = \|x\|^2 + \|y\|^2$

You Try It!

Example: Which of the following pairs of vectors are orthogonal? Find all pairs.

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_5 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

Orthogonality
Example

Problem: Find *all* vectors orthogonal to $v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

Key insight:

$$0 = x \cdot v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = x_1 + x_2 - x_3.$$

Orthogonality
Example

Problem: Find *all* vectors orthogonal to both $v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Key insight:

$$0 = x \cdot v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = x_1 + x_2 - x_3$$

$$0 = x \cdot w = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = x_1 + x_2 + x_3.$$

Orthogonality
General procedure

Problem: Find all vectors orthogonal to some number of vectors $v_1, v_2, \ldots, v_m$ in $\mathbf{R}^n$.

This is the same as finding all vectors x such that

$$0 = v_1^T x = v_2^T x = \cdots = v_m^T x.$$

Putting the *row* vectors $v_1^T, v_2^T, \ldots, v_m^T$ into a matrix, this is the same as finding all x such that

$$\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix} x = \begin{pmatrix} v_1 \cdot x \\ v_2 \cdot x \\ \vdots \\ v_m \cdot x \end{pmatrix} = 0.$$

Important

The set of all vectors orthogonal to some vectors $v_1, v_2, \ldots, v_m$ in $\mathbf{R}^n$ is the *null space* of the $m \times n$ matrix you get by "turning them sideways and smooshing them together:"

$$\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}.$$

In particular, this set is a subspace!

Summary

- The **dot product** of vectors x, y in $\mathbf{R}^n$ is the number $x^T y$.
- The **length** or **norm** of a vector x in $\mathbf{R}^n$ is $\|x\| = \sqrt{x \cdot x}$.
- The **distance** between two vectors x, y in $\mathbf{R}^n$ is $\text{dist}(x, y) = \|y - x\|$.
- A **unit vector** is a vector v with length $\|v\| = 1$.
- The **unit vector in the direction of** x is $x/\|x\|$.
- Two vectors x, y are **orthogonal** if $x \cdot y = 0$.
- The set of all vectors orthogonal to some vectors $v_1, v_2, \ldots, v_m$ in $\mathbf{R}^n$ is the null space of the matrix

$$\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}.$$