Section 1.3

Parametric Form

Example 1

How do we solve a system of linear equations if the row reduced matrix has a column without a pivot? Let's do an example.

$$\begin{aligned} 2x + y + 12z &= 1 \\ x + 2y + 9z &= -1 \end{aligned} \quad \text{gives rise to the matrix} \quad \left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right).$$

https://textbooks.math.gatech.edu/ila/demos/rrinter.html?mat=2,1,12,1:1,2,9,-1

Free Variables

Definition

Consider a *consistent* linear system of equations in the variables $x_1, \ldots, x_n$. Let A be a row echelon form of the matrix for this system.

We say that x_i is a **free variable** if its corresponding column in A is *not* a pivot column.

Important

1. You can choose *any value* for the free variables in a (consistent) linear system.
2. Free variables come from *columns without pivots* in a matrix in row echelon form.

Example 2

Suppose the reduced row echelon form of the matrix for a linear system in x_1, x_2, x_3, x_4 is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right)$$

What happened to x_2? What is it allowed to be?

The boxed equation is called the **parametric form** of the general solution to the system of equations. It is obtained by moving all free variables to the right-hand side of the $=$.

Example 3

Solve the system of linear equations in x_1, x_2, x_3, x_4:

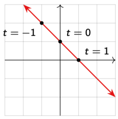
$$\begin{aligned} x_1 \quad + 5x_3 \quad &= 0 \\ x_4 &= 0 \end{aligned}$$

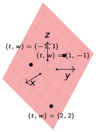
Free variables

Geometry, looking ahead!

If we have a consistent system of linear equations, with n variables and k free variables, then the set of solutions is a k-dimensional object in $\mathbf{R}^n$. We will make this precise later, but it is worth thinking about now.

Why does this make sense?





You try it!

Poll

A linear system has 4 variables and 3 equations.

What are the possible solution sets?

(a) point
(b) two points
(c) line
(d) plane
(e) 3-dimensional object
(f) 4-dimensional object

Yet Another Example

The linear system

$$x + y + z = 1 \quad \text{has matrix form} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right).$$

Q: What is the parametric form of the general solution to this linear system of one equation in three variables?

Trichotomy

There are *three possibilities* for the reduced row echelon form of the augmented matrix of a linear system.

1. The last column is a pivot column.
   In this case, the system is *inconsistent*. There are *zero* solutions, i.e. the solution set is *empty*. Picture:

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

2. Every column except the last column is a pivot column.
   In this case, the system has a *unique solution*. Picture:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$

3. The last column is not a pivot column, and some other column isn't either.
   In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). Picture:

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

Summary

- **Row reduction** is an algorithm for solving a system of linear equations represented by an augmented matrix.
- The goal of row reduction is to put a matrix into **(reduced) row echelon form**, which is the "solved" version of the matrix.
- An augmented matrix corresponds to an inconsistent system if and only if there is a pivot in the augmented column.
- Columns without pivots in the RREF of a matrix correspond to **free variables**. You can assign any value you want to the free variables.
- We can tell whether a linear system has zero, one, or infinitely many solutions using the RREF of the augmented matrix.

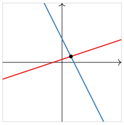
Chapter 2

Systems of Linear Equations: Geometry

Motivation

We want to think about the *algebra* in linear algebra (systems of equations and their solution sets) in terms of *geometry* (points, lines, planes, etc).

$$\begin{aligned} x - 3y &= -3 \\ 2x + y &= 8 \end{aligned} \quad \rightsquigarrow$$



Section 2.1

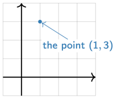
Vectors

Points and Vectors

We have been drawing elements of $\mathbf{R}^n$ as points in the line, plane, space, etc. We can also draw them as arrows.

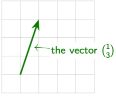
Definition

A **point** is an element of $\mathbf{R}^n$, drawn as a point (a dot).



A **vector** is an element of $\mathbf{R}^n$, drawn as an arrow. When we think of an element of $\mathbf{R}^n$ as a vector, we'll usually write it vectically, like a matrix with one column:

$$v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$



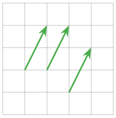
[interactive]

The difference is purely psychological: *points and vectors are just lists of numbers.*

Points and Vectors

So why make the distinction?

A vector need not start at the origin: *it can be located anywhere*! In other words, an arrow is determined by its length and its direction, not by its location.



These arrows all represent the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

> However, unless otherwise specified, we'll assume a vector starts at the origin.

Vector Algebra

Definition

- We can add two vectors together:

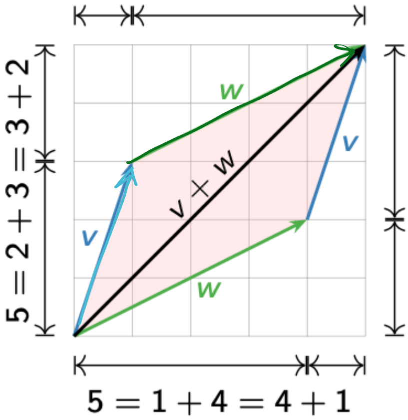
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix}.$$

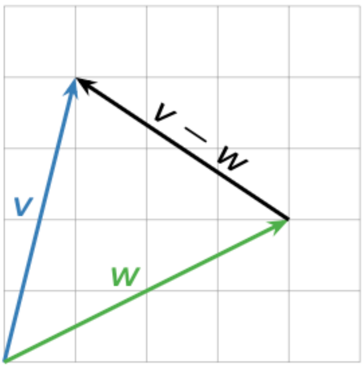
- We can multiply, or **scale**, a vector by a real number c:

$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \cdot x \\ c \cdot y \\ c \cdot z \end{pmatrix}.$$

We call c a **scalar** to distinguish it from a vector. If v is a vector and c is a scalar, cv is called a **scalar multiple** of v.

Vector Addition and Subtraction: Geometry

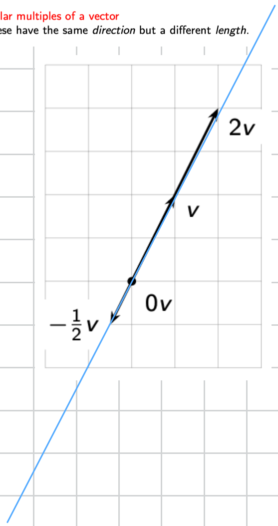




Scalar Multiplication: Geometry

Scalar multiples of a vector

These have the same *direction* but a different *length*.



Linear Combinations

We can add and scalar multiply in the same equation:

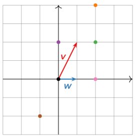
$$w = c_1v_1 + c_2v_2 + \cdots + c_pv_p$$

where $c_1, c_2, \ldots, c_p$ are scalars, $v_1, v_2, \ldots, v_p$ are vectors in $\mathbf{R}^n$, and w is a vector in $\mathbf{R}^n$.

Definition

We call w a **linear combination** of the vectors $v_1, v_2, \ldots, v_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the **weights** or **coefficients**.

Example



Let $v = \begin{pmatrix}1\\2\end{pmatrix}$ and $w = \begin{pmatrix}1\\0\end{pmatrix}$.

What are some linear combinations of v and w?

- $v + w$
- $v - w$
- $2v + 0w$
- $2w$
- $-v$

[interactive: 2 vectors] [interactive: 3 vectors]

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=1,2&v2=1,0&range=5&captions=combo&nomove=true&labels=v,w

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,1,-1&v3=-1,1,4&range=5&captions=combo&nomove=true

You Try It!

Example. If v=[1;2] and w=[1;0] then find

- v+w
- v-w
- 2v+0w
- 2w
- -v

Poll

Poll

Is there any vector in $\mathbf{R}^2$ that is *not* a linear combination of v and w?

Poll

Poll

Is there any vector in $\mathbf{R}^2$ that is *not* a linear combination of v and w?

More Examples

What are some linear combinations of $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

What are some linear combinations of $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

Section 2.2

Vector Equations and Spans

Systems of Linear Equations

Solve the following system of linear equations:

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3. \end{aligned}$$

We can write all three equations at once as vectors:

$$\begin{pmatrix} x - y \\ 2x - 2y \\ 6x - y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

We can write this as a linear combination:

$$x \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

So we are asking:

Question: Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

Summary

The **vector equation**

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b,$$

where $v_1, v_2, \ldots, v_p, b$ are vectors in $\mathbf{R}^n$ and $x_1, x_2, \ldots, x_p$ are scalars, has the same solution set as the linear system with augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array}\right),$$

where the v_i's and b are the columns of the matrix.

This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

"the set of" "such that"

Definition

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$. In symbols:

$$\longrightarrow \boxed{\mathrm{Span}\{v_1, v_2, \ldots, v_p\} = \left\{ x_1 v_1 + x_2 v_2 + \cdots + x_p v_p \mid x_1, x_2, \ldots, x_p \text{ in } \mathbf{R} \right\}.}$$

Synonyms: $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is the subset **spanned by** or **generated by** $v_1, v_2, \ldots, v_p$.

Now we have several equivalent ways of making the same statement:

1. A vector b is in the span of $v_1, v_2, \ldots, v_p$.
2. The vector equation
$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$
has a solution.
3. The linear system with augmented matrix
$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array}\right)$$
is consistent.

Example: Is the vector b in the span of v and w, where b=[1;2;1], v=[1;0;-1] and w=[0;1;1]?

You Try It!

Poll

Poll

How many vectors are in Span $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$?

A. Zero
B. One
C. Infinity

Example: If v1=[2;1], what is Span{v1,3v1}?

Example: Solve the system and write out the corresponding vector equation. How can the system be interpreted geometrically?

x-y=8
2x-2y=16
6x-y=3

Example: What is Span{[1;-1;0],[2;7;0],[10;-6;0]}?

Summary

The whole lecture was about drawing pictures of systems of linear equations.

- **Points** and **vectors** are two ways of drawing elements of $\mathbf{R}^n$. Vectors are drawn as arrows.
- Vector addition, subtraction, and scalar multiplication have geometric interpretations.
- A **linear combination** is a sum of scalar multiples of vectors. This is also a geometric construction, which leads to lots of pretty pictures.
- The **span** of a set of vectors is the set of all linear combinations of those vectors. It is also fun to draw.
- A system of linear equations is equivalent to a vector equation, where the unknowns are the coefficients of a linear combination.