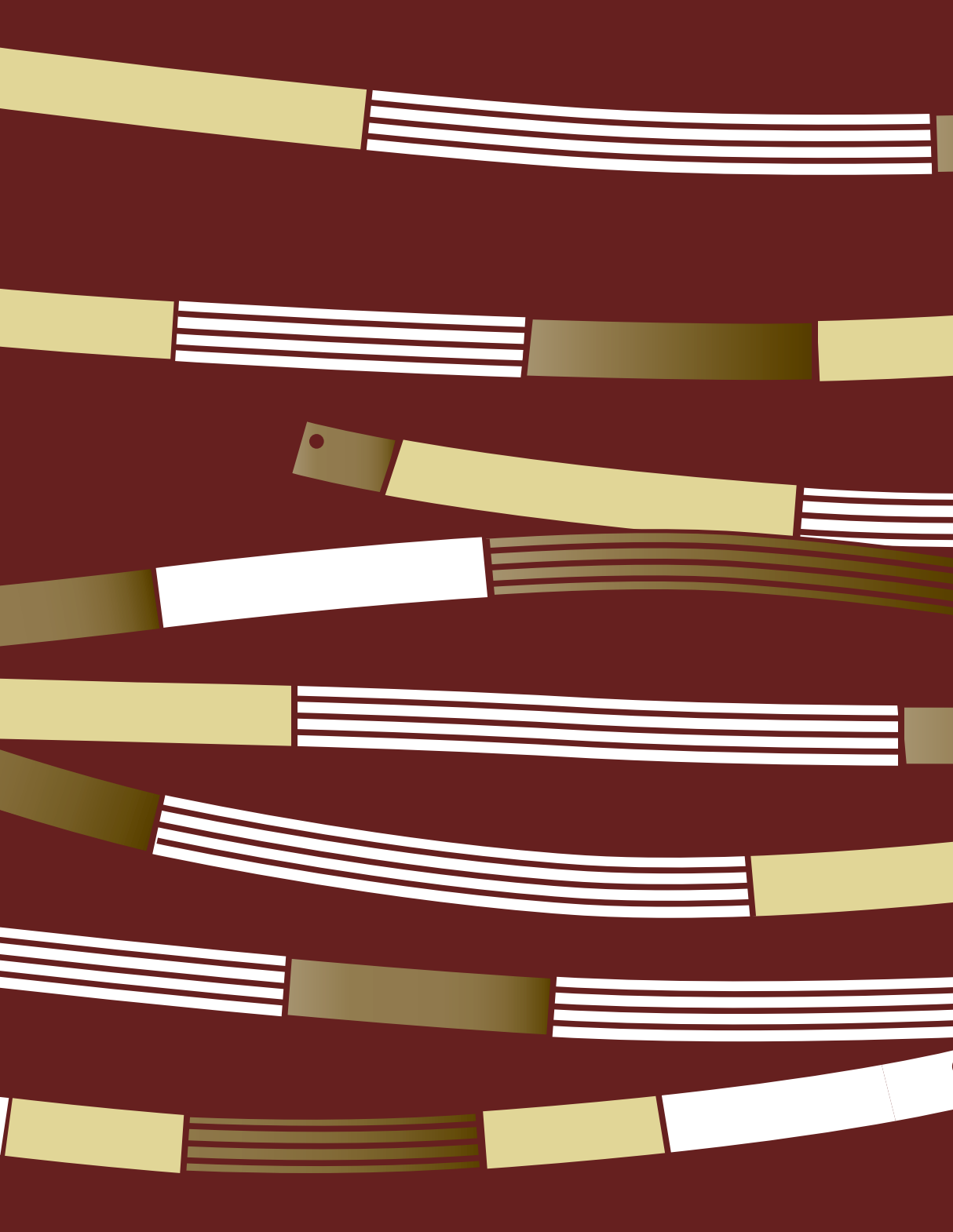




Chapter 2

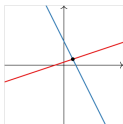
Systems of Linear Equations: Geometry

Motivation

We want to think about the *algebra* in linear algebra (systems of equations and their solution sets) in terms of *geometry* (points, lines, planes, etc).

$$\begin{aligned}x - 3y &= -3 \\ 2x + y &= 8\end{aligned}$$

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## Section 2.1

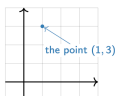
### Vectors

#### Points and Vectors

We have been drawing elements of  $\mathbb{R}^n$  as points in the line, plane, space, etc. We can also draw them as arrows.

##### Definition

A **point** is an element of  $\mathbb{R}^n$ , drawn as a point (a dot).



A **vector** is an element of  $\mathbb{R}^n$ , drawn as an arrow. When we think of an element of  $\mathbb{R}^n$  as a vector, we'll usually write it vertically, like a matrix with one column:

$$v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$



[interactive]

The difference is purely psychological: *points and vectors are just lists of numbers.*

#### Points and Vectors

So why make the distinction?

A vector need not start at the origin: *it can be located anywhere!* In other words, an arrow is determined by its length and its direction, not by its location.



These arrows all represent the vector  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .

However, unless otherwise specified, we'll assume a vector starts at the origin.

#### Vector Algebra

##### Definition

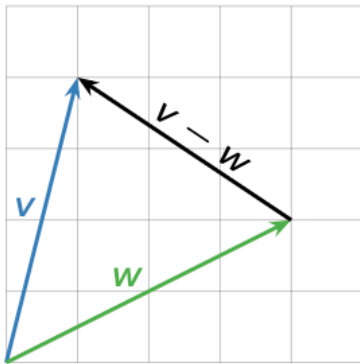
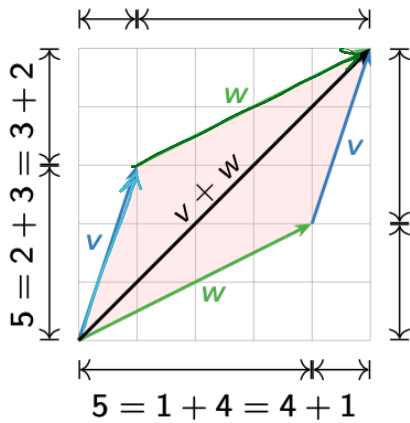
► We can add two vectors together:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix}.$$

► We can multiply, or **scale**, a vector by a real number  $c$ :

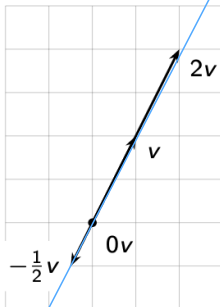
$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \cdot x \\ c \cdot y \\ c \cdot z \end{pmatrix}.$$

We call  $c$  a **scalar** to distinguish it from a vector. If  $v$  is a vector and  $c$  is a scalar,  $cv$  is called a **scalar multiple** of  $v$ .



### Scalar Multiplication: Geometry

**Scalar multiples of a vector**  
These have the same *direction* but a different *length*.



## Linear Combinations

We can add and scalar multiply in the same equation:

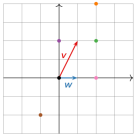
$$w = c_1v_1 + c_2v_2 + \dots + c_pv_p$$

where  $c_1, c_2, \dots, c_p$  are scalars,  $v_1, v_2, \dots, v_p$  are vectors in  $\mathbf{R}^n$ , and  $w$  is a vector in  $\mathbf{R}^n$ .

### Definition

We call  $w$  a **linear combination** of the vectors  $v_1, v_2, \dots, v_p$ . The scalars  $c_1, c_2, \dots, c_p$  are called the **weights** or **coefficients**.

### Example



Let  $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and  $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

What are some linear combinations of  $v$  and  $w$ ?

- $v + w$
- $v - w$
- $2v + 0w$
- $2w$
- $-v$

[interactive: 2 vectors]

[interactive: 3 vectors]

<https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=1,2&v2=1,0&range=5&captions=combo&nomove=true&labels=v,w>

<https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,1&v3=-1,1,4&range=5&captions=combo&nomove=true>

### You Try It!

Example. If  $v = [1; 2]$  and  $w = [1; 0]$  then find

- $v + w$
- $v - w$
- $2v + 0w$
- $2w$
- $-v$

## Poll

### Poll

Is there any vector in  $\mathbf{R}^2$  that is *not* a linear combination of  $v$  and  $w$ ?

## Poll

Poll

Is there any vector in  $\mathbb{R}^2$  that is *not* a linear combination of  $v$  and  $w$ ?

## More Examples

What are some linear combinations of  $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ?

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

What are some linear combinations of  $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ?

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

## Section 2.2

### Vector Equations and Spans

#### Systems of Linear Equations

Solve the following system of linear equations:

$$\begin{aligned}x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3.\end{aligned}$$

We can write all three equations at once as vectors:

$$\begin{pmatrix} x & -y \\ 2x & -2y \\ 6x & -y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

We can write this as a linear combination:

$$x \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

So we are asking:

**Question:** Is  $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$  a linear combination of  $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$  and  $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$ ?

#### Summary

##### The vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b,$$

where  $v_1, v_2, \dots, v_p, b$  are vectors in  $\mathbb{R}^n$  and  $x_1, x_2, \dots, x_p$  are scalars, has the same solution set as the linear system with augmented matrix

$$\left( \begin{array}{c|c|c|c|c} | & | & \cdots & | & | \\ v_1 & v_2 & & v_p & b \\ | & | & & | & | \end{array} \right),$$

where the  $v_i$ 's and  $b$  are the columns of the matrix.

This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

#### Definition

Let  $v_1, v_2, \dots, v_p$  be vectors in  $\mathbb{R}^n$ . The **span** of  $v_1, v_2, \dots, v_p$  is the collection of all linear combinations of  $v_1, v_2, \dots, v_p$ , and is denoted  $\text{Span}\{v_1, v_2, \dots, v_p\}$ . In symbols:

$$\text{Span}\{v_1, v_2, \dots, v_p\} = \{x_1 v_1 + x_2 v_2 + \cdots + x_p v_p \mid x_1, x_2, \dots, x_p \text{ in } \mathbb{R}\}.$$

**Synonyms:**  $\text{Span}\{v_1, v_2, \dots, v_p\}$  is the subset **spanned by** or **generated by**  $v_1, v_2, \dots, v_p$ .

Now we have several equivalent ways of making the same statement:

1. A vector  $b$  is in the span of  $v_1, v_2, \dots, v_p$ .
2. The vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$

has a solution.

3. The linear system with augmented matrix

$$\left( \begin{array}{c|c|c|c|c} | & | & \cdots & | & | \\ v_1 & v_2 & & v_p & b \\ | & | & & | & | \end{array} \right)$$

is consistent.

**Example:** Is the vector  $b$  in the span of  $v$  and  $w$ , where  $b=[1;2;1]$ ,  $v=[1;0;-1]$  and  $w=[0;1;1]$ ?

## You Try It!

### Poll

#### Poll

How many vectors are in  $\text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ ?

- A. Zero
- B. One
- C. Infinity

Example: If  $v_1 = [2; 1]$ , what is  $\text{Span}\{v_1, 3v_1\}$ ?

Example: Solve the system and write out the corresponding vector equation. How can the system be interpreted geometrically?

$$\begin{aligned}x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3\end{aligned}$$

Example: What is  $\text{Span}\{[1;-1;0],[2;7;0],[10;-6;0]\}$ ?

#### Summary

The whole lecture was about drawing pictures of systems of linear equations.

- ▶ **Points and vectors** are two ways of drawing elements of  $\mathbb{R}^n$ . Vectors are drawn as arrows.
- ▶ Vector addition, subtraction, and scalar multiplication have geometric interpretations.
- ▶ A **linear combination** is a sum of scalar multiples of vectors. This is also a geometric construction, which leads to lots of pretty pictures.
- ▶ The **span** of a set of vectors is the set of all linear combinations of those vectors. It is also fun to draw.
- ▶ A system of linear equations is equivalent to a vector equation, where the unknowns are the coefficients of a linear combination.



