



Sections 2.7 and 2.9

Basis, Dimension, Rank and Basis Theorems

Subspaces

Reminder

Recall: a subspace of $\mathbb{R}^n$ is the same thing as a span, except we haven't computed a spanning set yet.

For example, $\text{Col } A$ and $\text{Nul } A$ for a matrix A .

There are lots of choices of spanning set for a given subspace.

Are some better than others?

how you choose
to represent a subspace
as a span of vectors
is unique!

Basis of a Subspace

What is the *smallest number* of vectors that are needed to span a subspace?

Definition

Let V be a subspace of $\mathbb{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \dots, v_m\}$ in V such that:

1. $V = \text{Span}\{v_1, v_2, \dots, v_m\}$, and
2. $\{v_1, v_2, \dots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V , and is written $\dim V$.

Note the big red border here

Important

A subspace has *many different* bases, but they all have the same number of vectors.

Bases of $\mathbb{R}^2$

Question

What is a basis for $\mathbb{R}^2$?



✓

✓

Fact
 $\mathbb{R}^n$ has dimension equal to n .
So any basis for $\mathbb{R}^n$ has exactly n vectors in it.
e.g. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Bases of $\mathbb{R}^n$

The unit coordinate vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

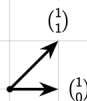
are a basis for $\mathbb{R}^n$. The identity matrix has columns $e_1, e_2, \dots, e_n$.

1. They span: I_n has a pivot in every row.
2. They are linearly independent: I_n has a pivot in every column.

In general: $\{v_1, v_2, \dots, v_n\}$ is a basis for $\mathbb{R}^n$ if and only if the matrix

$$A = \left(\begin{array}{c|c|c|c} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{array} \right) \begin{matrix} \text{size} \\ n \times n \end{matrix}$$

has a pivot in every row and every column.



Basis of a Subspace

Example

Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbb{R}^3 \mid x + 3y + z = 0 \right\} \quad B = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$

Verify that B is a basis for V . (So $\dim V = 2$: it is a *plane*.)

<https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=-3,1,0&v2=0,1,-3&range=5&captions=combo>

Q: What is another basis
for $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + 3y + z = 0 \right\}$

The set V is just $\underline{\hspace{2cm}}$ to the system $x + 3y + z = 0$.

Example: Find a basis for $\text{Nul}(A)$ with A given below
(the RREF of A is also given)

Basis for $\text{Nul}(A)$

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Fact

The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for $\text{Nul } A$.

Example: Find a basis for $\text{Nul}(A)$ with A given below
(the RREF of A is also given)

Basis for $\text{Nul } A$

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \quad \xrightarrow{\text{rref}} \quad \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Fact

The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for $\text{Nul } A$.

Summary so far.

A basis for subspace V is a set of vectors $\{v_1, v_2, v_3\}$ s.t.

- ① $\{v_1, v_2, v_3\}$ span the subspace V
- ② $\{v_1, v_2, v_3\}$ are lin. ind.

Example: find a basis for $\text{Col}(A)$ for the matrix A given below (the RREF of A is also given)

Basis for $\text{Col } A$

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis $\longleftrightarrow$ pivot columns in rref

Fact

The pivot columns of A always form a basis for $\text{Col } A$.

Warning: I mean the pivot columns of the original matrix A , not the row-reduced form. (Row reduction changes the column space.)

(why 2?)

NOTE

$$\text{Col } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} \right\}$$

$\uparrow$ in general $A = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$ (size 3×4)

Then $\text{Col } A$ could be (a priori)

(1) only $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

(2) a line in $\mathbb{R}^3$

(3) a plane in $\mathbb{R}^3$

(3) all of $\mathbb{R}^3$

IDEA: just extract the pivot cols of the ORIGINAL MATRIX.

$\&$ These will be a basis for $\text{Col } A$.

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis $\longleftrightarrow$ pivot columns in rref

The Basis Theorem

Basis Theorem

Let V be a subspace of dimension m . Then:

- Any m linearly independent vectors in V form a basis for V .
- Any m vectors that span V form a basis for V .

Example: any three linearly independent vectors form a basis for $\mathbb{R}^3$.

Upshot

If you already know that $\dim V = m$, and you have m vectors $B = \{v_1, v_2, \dots, v_m\}$ in V , then you only have to check one of

- B is linearly independent, or
- B spans V

in order for B to be a basis.

The Rank Theorem

Recall:

- ▶ The **dimension** of a subspace V is the number of vectors in a basis for V .
- ▶ A basis for the column space of a matrix A is given by the pivot columns.
- ▶ A basis for the null space of A is given by the vectors attached to the free variables in the parametric vector form.

Definition

The **rank** of a matrix A , written $\text{rank } A$, is the dimension of the column space $\text{Col } A$. The **nullity** of A , written $\text{nullity } A$, is the dimension of the solution set of $Ax = 0$.

$\uparrow$ $\text{rank}(A) = \# \text{ pivot} = \dim \text{Col } A$.
 $\text{nullity}(A) = \# \text{ free vars} = \dim \text{Nul } A$.

← NOT the columns
 or the REF/REF
 (you'll have the right number of
 vectors but the wrong span!)

Rank Theorem

If A is an $m \times n$ matrix, then

$$\text{rank } A + \text{nullity } A = n = \text{the number of columns of } A.$$

so that is:

$$(\text{dimension of column space}) + (\text{dimension of solution set to } Ax=0) = (\text{number of variables})$$

The Rank Theorem

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

basis of Col A (points to pivot columns 1 and 2)
 free variables (points to columns 3 and 4)

You Try It!

Poll

Poll

True or False: If A is a 10×15 matrix and there is a basis of $\text{Col } A$ consisting of 4 vectors, then there is a basis of $\text{Nul } A$ consisting of 6 vectors.

Summary

- ▶ A **basis** of a subspace is a minimal set of spanning vectors.
- ▶ There are recipes for computing a basis for the column space and null space of a matrix.
- ▶ The **dimension** of a subspace is the number of vectors in any basis.
- ▶ The **basis theorem** says that if you already know that $\dim V = n$, and you have m vectors in V , then you only have to check if they span or they're linearly independent to know they're a basis.
- ▶ The **rank theorem** says the dimension of the column space of a matrix, plus the dimension of the null space, is the number of columns of the matrix.

→ NOT THE BASIS

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis $\longleftrightarrow$ pivot columns in rref

extract THESE columns for
a basis for Col A

Not these columns!

Ex. Find a basis for $\text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix} \right\}$

META: ① write $A = [v_1 \ v_2 \ v_3] = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

② row reduce to REF (don't need RREF) to find pivots

③ extract pivot cols of ORIGINAL MATRIX (not the REF/RREF!)

Chapter 3

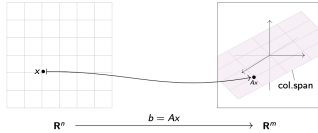
Linear Transformations and Matrix Algebra

Section 3.1

Matrix Transformations

Motivation

Let A be a matrix, and consider the matrix equation $b = Ax$. If we vary x , we can think of this as a function of x .



Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, plot some inputs x and outputs Ax

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \textcircled{1} \text{ For some } x \text{ in } \mathbb{R}^3, \text{ compute } Ax.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} =$$

Example: with A as below, plot some inputs and outputs

Matrices as Functions
Projection

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now find some "random" inputs x in $\mathbb{R}^3$ & compute the output Ax .

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{bmatrix} 2 + 0 + 0 \\ 0 + 3 + 0 \\ 0 + 0 + 0 \end{bmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} =$$

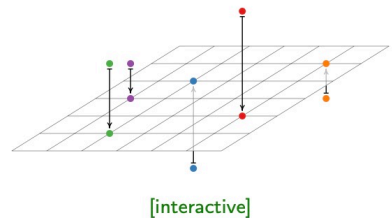
Good news & bad news:

In this class there are (at least) TWO different reasons for a topic/concept to be hard:

- ① Computations / technical difficulty
- ② connections between ideas / theoretical difficulty

[https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?](https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?mat=1,0,0:0,1,0:0,0,0&range2=5&closed=true)

[mat=1,0,0:0,1,0:0,0,0&range2=5&closed=true](https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?mat=1,0,0:0,1,0:0,0,0&range2=5&closed=true)



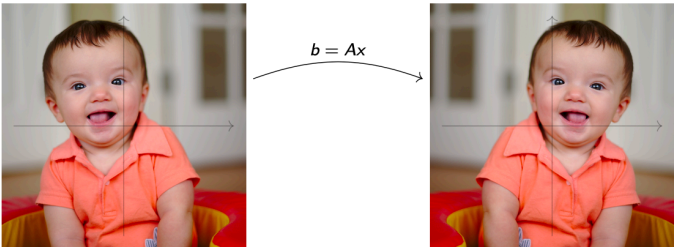
Example: for the matrix A given below, plot some input and output vectors for $T(x)=Ax$, and give the function T an appropriate NAME.

Matrices as Functions
Reflection

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

o

<https://textbooks.math.gatech.edu/ila/demos/twobytwo.html?mat=-1,0,0,1&closed=true>



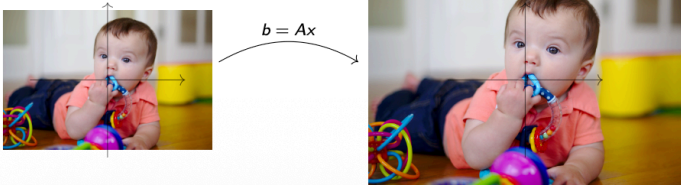
[interactive]

Example: plot some input and output vectors for $T(x)=Ax$ and give the function T a NAME.

Matrices as Functions
Dilation

$$A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}$$

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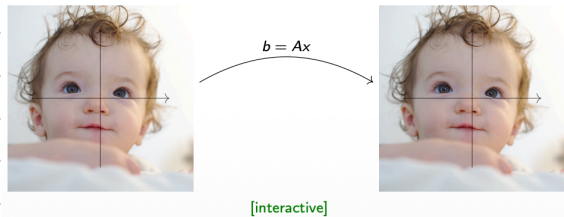
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Example: ditto

Matrices as Functions
Identity

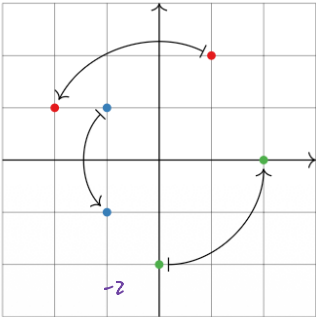
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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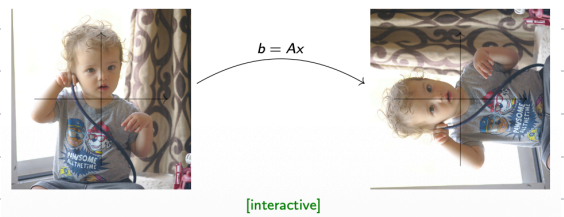


Matrices as Functions
Rotation

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



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Other Geometric Transformations

In §3.1 there are other examples of geometric transformations of $\mathbb{R}^2$ given by matrices. Please look them over.

Transformations

Motivation

We have been drawing pictures of what it looks like to multiply a matrix by a vector, as a function of the vector.

Now let's go the other direction. Suppose we have a function, and we want to know, does it come from a matrix?

Example

For a vector x in $\mathbb{R}^2$, let $T(x)$ be the counterclockwise rotation of x by an angle θ . Is $T(x) = Ax$ for some matrix A ?

If $\theta = 90^\circ$, then we know $T(x) = Ax$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

But for general θ , it's not clear.

Our next goal is to answer this kind of question.

The matrix A is doing something
to input vectors to get output vectors

Transformations

Vocabulary

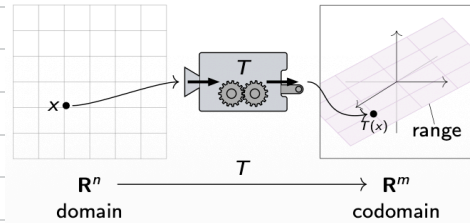
Definition

A **transformation** (or **function** or **map**) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule T that assigns to each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$.

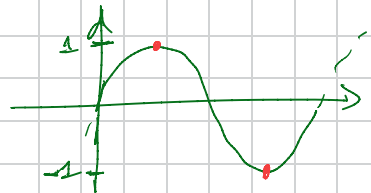
- $\mathbb{R}^n$ is called the **domain** of T (the inputs).
- $\mathbb{R}^m$ is called the **codomain** of T (where the outputs live).
- For x in $\mathbb{R}^n$, the vector $T(x)$ in $\mathbb{R}^m$ is the **image** of x under T .
- Notation:** $x \mapsto T(x)$.
- The set of all images $\{T(x) \mid x \text{ in } \mathbb{R}^n\}$ is the **range** of T .

Notation:

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ means T is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.



Examples from calculus: $f(x)=\sin(x)$ and $g(x)=x^2$



Functions from Engineering

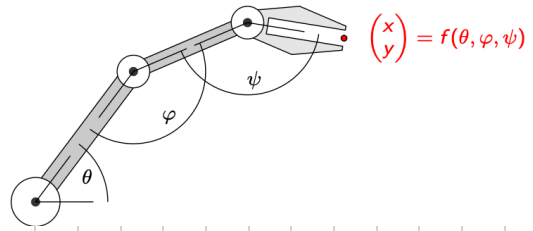
Suppose you are building a robot arm with three joints that can move its hand around a plane, as in the following picture.

Matrix Transformations

Definition

Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \text{defined by} \quad T(x) = Ax.$$



Example: find the domain, codomain, and range of the matrix transformation $T(x)$ below

$$T \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -3 & -3 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -14 \\ -32 \\ -14 \end{pmatrix}.$$

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

- ① domain is $\mathbb{R}^3$
- ② codomain is $\mathbb{R}^3$
- ③ range is ???

where inputs live

- ▶ The *domain* of T is $\mathbb{R}^n$, which is the number of *columns* of A .
- ▶ The *codomain* of T is $\mathbb{R}^m$, which is the number of *rows* of A .
- ▶ The *range* of T is the set of all images of T :

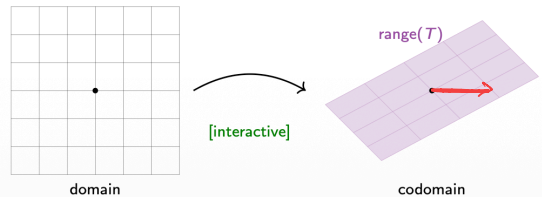
$$T(x) = Ax = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

where outputs live

Example: find the domain, codomain, and range of the matrix transformation $T(x)=Ax$ with the matrix A given below

$$A = \begin{pmatrix} -1 & 0 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$$

<https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?mat=-1,0,2,1;-1,1&closed=true&show=false>



You Try It!

Poll

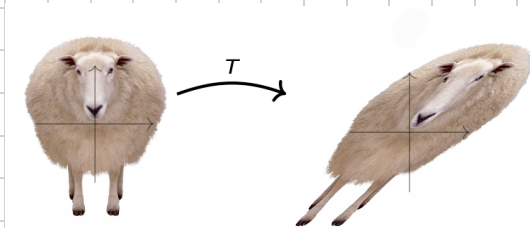
Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$. (T is called a **shear**.)

Find: the domain, the codomain, and the range of T .

Also plot a few input and output vectors, and give the transformation a NAME

Summary

- ▶ We can think of $b = Ax$ as a **transformation** with input x and output b .
- ▶ There are vocabulary words associated to transformations: **domain**, **codomain**, **range**.
- ▶ A transformation that comes from a matrix is called a **matrix transformation**.
- ▶ In this case, the vocabulary words mean something concrete in terms of matrices.
- ▶ We like transformations that come from matrices, because questions about those transformations turn into questions about matrices.



Section 3.2

One-to-one and Onto Transformations

Matrix Transformations

Reminder

Recall: Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \text{defined by} \quad T(x) = Ax.$$

- ▶ The *domain* of T is $\mathbb{R}^n$, which is the number of *columns* of A .
- ▶ The *codomain* of T is $\mathbb{R}^m$, which is the number of *rows* of A .
- ▶ The *range* of T is the set of all images of T :

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

This is the *column space* of A . It is a span of vectors in the codomain.

4

Matrix Transformations

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

(1) find $T(u)$ where $u = [3; 4]$

(2) let $b = [7; 5; 7]$, find v in $\mathbb{R}^2$ such that $T(v) = b$.

(3) is there any other vector w (other than v) that also gets sent to b ?

1,

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

Same A as previous example.

(4) is there any vector c in $\mathbb{R}^3$ which is not in the range of T ? In other words, can you find a vector c for which NO vector v in $\mathbb{R}^2$ maps to c ?

Translation: Find c such that $Ax = c$ is inconsistent.

Matrix Transformations

Non-Example

Note: All of these questions are questions about the transformation T ; it still makes sense to ask them in the absence of the matrix A .

The fact that T comes from a matrix means that these questions translate into questions about a matrix, which we know how to do.

Non-example: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$

We will restrict our focus to matrix transformations in this class, but it is good to know that you can still ask similar questions about arbitrary transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Questions About Transformations

Today we will focus on two important questions one can ask about a transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$:

- Do there exist distinct vectors x, y in $\mathbb{R}^n$ such that $T(x) = T(y)$?
- For every vector v in $\mathbb{R}^m$, does there exist a vector x in $\mathbb{R}^n$ such that $T(x) = v$?

These are subtle because of the multiple *quantifiers* involved ("for every", "there exists").

One-to-one Transformations

Definition

A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one-to-one** (or **into**, or **injective**) if different vectors in $\mathbb{R}^n$ map to different vectors in $\mathbb{R}^m$. In other words, for every b in $\mathbb{R}^m$, the equation $T(x) = b$ has at most one solution x . Or, different inputs have different outputs. Note that *not one-to-one* means at least two different vectors in $\mathbb{R}^n$ have the same image.

Matrix Transformations

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

Same example as before: Is the transformation $T(x)=Ax$ one-to-one?

Characterization of One-to-One Matrix Transformations

Theorem

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a matrix transformation with matrix A . Then the following are equivalent:

- T is one-to-one.
- For each b in $\mathbb{R}^m$, the equation $T(x) = b$ has at most one solution.
- For each b in $\mathbb{R}^m$, the equation $Ax = b$ has a unique solution or is inconsistent.
- $Ax = 0$ has a unique solution.
- The columns of A are linearly independent.
- A has a pivot in every column.

Question

If $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one, what can we say about the relative sizes of n and m ?

What can you say about A ? $n \leq m$ (#cols $\leq$ #rows)

What sizes can A be? can be TALL

What sizes can A NOT be? can NOT be WIDE

New example:

One-to-One Transformations Example

Define

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad T(x) = Ax,$$

so $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$. Is T one-to-one?

- (1) domain of T is
- (2) codomain of T is
- (3) range of T is a line in $\mathbb{R}^3$
- (4) is T one-to-one?

Non-example:

One-to-One Transformations Non-Example

Define

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad T(x) = Ax,$$

- (1) Is the transformation $T(x)=Ax$ one-to-one?
- (2) If not, find two **different** vectors v and w that map to the **same** output b
- (3) Also, is **every** vector b in $\mathbb{R}^2$ the output for some input vector v ?

One-to-one Transformations

Definition

A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one-to-one** (or **into**, or **injective**) if different vectors in $\mathbb{R}^n$ map to different vectors in $\mathbb{R}^m$. In other words, for every b in $\mathbb{R}^m$, the equation $T(x) = b$ has at *most* one solution x . Or, different inputs have different outputs. Note that *not* one-to-one means at least two different vectors in $\mathbb{R}^n$ have the same image.

Onto Transformations

Definition

A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **onto** (or **surjective**) if the range of T is equal to $\mathbb{R}^m$ (its codomain). In other words, for every b in $\mathbb{R}^m$, the equation $T(x) = b$ has at least one solution. Or, every possible output has an input. Note that not onto means there is some b in $\mathbb{R}^m$ which is not the image of any x in $\mathbb{R}^n$.

Same example as before:

One-to-One Transformations

Non-Example

Define

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad T(x) = Ax,$$

(1) is $T(x) = Ax$ onto?

Characterization of Onto Matrix Transformations

Theorem

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a matrix transformation with matrix A . Then the following are equivalent:

- ▶ T is onto
- ▶ $T(x) = b$ has a solution for every b in $\mathbb{R}^m$
- ▶ $Ax = b$ is consistent for every b in $\mathbb{R}^m$
- ▶ The columns of A span $\mathbb{R}^m$
- ▶ A has a pivot in every row

Same example as before:

Matrix Transformations

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

(1) is $T(x) = Ax$ onto?

Question

If $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is onto, what can we say about the relative sizes of n and m ?

* IF A is $m \times n$ then $m \leq n$ (# rows $\leq$ # cols)

* A can be WIDE for T to be onto

* A can NOT be tall for T to be onto.

Last example:

One-to-One and Onto Transformations

Non-Example

Define

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad T(x) = Ax,$$

so $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$. Is T one-to-one? Is it onto?

(1) find the domain and codomain of T

(2) is T one-to-one?

(3) is T onto?

Summary

- ▶ A transformation T is **one-to-one** if $T(x) = b$ has at *most one* solution, for every b in $\mathbb{R}^n$.
- ▶ A transformation T is **onto** if $T(x) = b$ has at *least one* solution, for every b in $\mathbb{R}^n$.
- ▶ A matrix transformation with matrix A is one-to-one if and only if the columns of A are linearly independent, if and only if A has a pivot in every column.
- ▶ A matrix transformation with matrix A is onto if and only if the columns of A span $\mathbb{R}^n$, if and only if A has a pivot in every row.
- ▶ Two of the most basic questions one can ask about a transformation is whether it is one-to-one or onto.