MA1553

Section 6.2

Orthogonal Complements

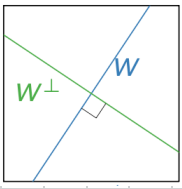
Orthogonal Complements

Definition
Let W be a subspace of $\mathbf{R}^n$. Its **orthogonal complement** is

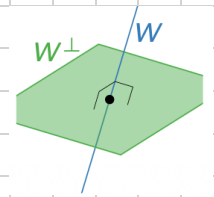
$$W^{\perp} = \{v \text{ in } \mathbf{R}^n \mid v \cdot w = 0 \text{ for all } w \text{ in } W\}$$

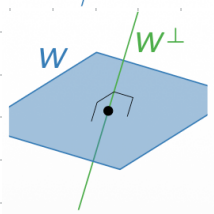
read "W perp".

$W^{\perp}$ is orthogonal complement
A^T is transpose



https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=.3,0,1&captions=orthog&range=3





You Try It!

Poll

Let W be a 2-plane in $\mathbf{R}^4$. How would you describe $W^{\perp}$?

A. The zero space $\{0\}$.
B. A line in $\mathbf{R}^4$.
C. A plane in $\mathbf{R}^4$.
D. A 3-dimensional space in $\mathbf{R}^4$.
E. All of $\mathbf{R}^4$.

Hint: try an example! What is the set of vectors which are orthogonal to both e1 and e2 in R^4?

For example, if W is the xy-plane, then $W^{\perp}$ is the zw-plane:

$$\begin{pmatrix} x \\ y \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ z \\ w \end{pmatrix} = 0.$$

Orthogonal Complements

Basic properties

Let W be a subspace of $\mathbf{R}^n$.

Facts:

1. $W^\perp$ is also a subspace of $\mathbf{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $A = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \end{pmatrix}$ and $W = \text{Col } A$, then $W^\perp = \text{Nul}(A^T)$

Problem: if $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, compute $W^\perp$.

Hint: use (4.)

(4.) says:

$$\text{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \text{Nul} \begin{pmatrix} — & v_1^T & — \\ — & v_2^T & — \\ & \vdots & \\ — & v_m^T & — \end{pmatrix}$$

Orthogonal Complements

Row space, column space, null space

Definition

The **row space** of an $m \times n$ matrix A is the span of the *rows* of A. It is denoted $\operatorname{Row} A$. Equivalently, it is the column space of A^T:

$$\operatorname{Row} A = \operatorname{Col} A^T.$$

Fact: $(\operatorname{Row} A)^\perp = \operatorname{Nul} A.$

$(\operatorname{Col} A)^\perp = \operatorname{Nul} A^T$

Dimension of the row space

Even though $\operatorname{Row}(A)$ lives in $\mathbf{R}^n$ and $\operatorname{Col}(A)$ lives in $\mathbf{R}^m$ if A is an $m \times n$ matrix, both subspaces have the same dimension.

Theorem

If A is an $m \times n$ matrix, then $\dim(\operatorname{Row} A) = \dim(\operatorname{Col} A)$.

Idea: the dimension of Row A is equal to the number of pivots of A, which is the dimension of the column space of A.