Chapter 2

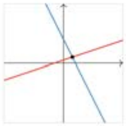
Systems of Linear Equations: Geometry

Motivation

We want to think about the *algebra* in linear algebra (systems of equations and their solution sets) in terms of *geometry* (points, lines, planes, etc).

algebraic

$$\begin{aligned} x - 3y &= -3 \\ 2x + y &= 8 \end{aligned} \quad \rightsquigarrow$$



vs. geometry.

Section 2.1

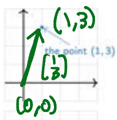
Vectors

Points and Vectors

We have been drawing elements of $\mathbf{R}^n$ as points in the line, plane, space, etc. We can also draw them as arrows.

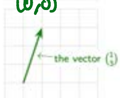
Definition

A **point** is an element of $\mathbf{R}^n$, drawn as a point (a dot).



A **vector** is an element of $\mathbf{R}^n$, drawn as an arrow. When we think of an element of $\mathbf{R}^n$ as a vector, we'll usually write it vectically, like a matrix with one column:

$$v = \begin{pmatrix}1\\3\end{pmatrix}.$$



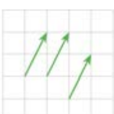
[interactive]

The difference is purely psychological: *points and vectors are just lists of numbers.*

Points and Vectors

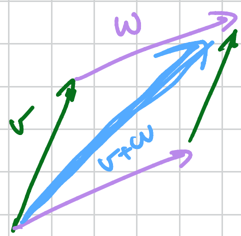
So why make the distinction?

A vector need not start at the origin: *it can be located anywhere!* In other words, an arrow is determined by its length and its direction, not by its location.



These arrows all represent the vector $\binom{1}{2}$.

However, unless otherwise specified, we'll assume a vector starts at the origin.



Vector Algebra

Definition

- We can add two vectors together:

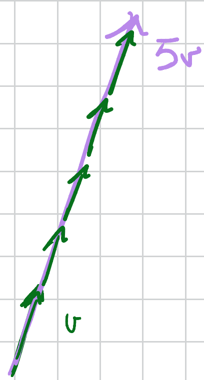
$$\begin{pmatrix}a\\b\\c\end{pmatrix} + \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a+x\\b+y\\c+z\end{pmatrix}.$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}-1\\2\\5\end{bmatrix} = \begin{bmatrix}0\\4\\8\end{bmatrix}$$

- We can multiply, or **scale**, a vector by a real number c:

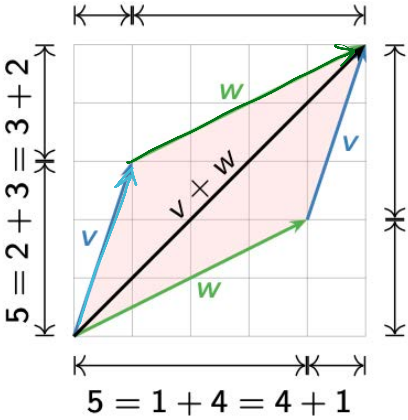
$$c\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}c\cdot x\\c\cdot y\\c\cdot z\end{pmatrix}.$$

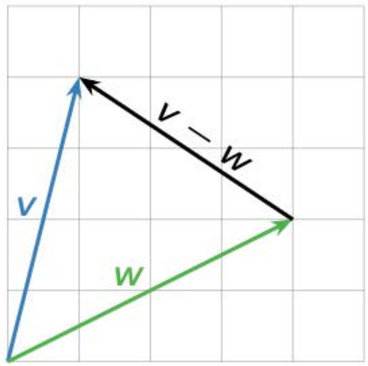
$$5\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}5\\10\\15\end{bmatrix}$$



We call c a **scalar** to distinguish it from a vector. If v is a vector and c is a scalar, cv is called a **scalar multiple** of v.

Vector Addition and Subtraction: Geometry



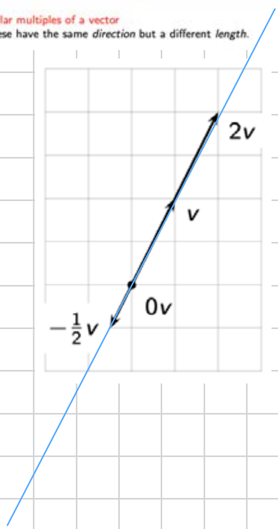


$\cancel{w} + (v - \cancel{w}) = v$

Scalar Multiplication: Geometry

Scalar multiples of a vector

These have the same *direction* but a different *length*.



Linear Combinations

We can add and scalar multiply in the same equation:

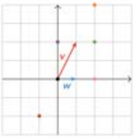
$$w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

where $c_1, c_2, \dots, c_p$ are scalars, $v_1, v_2, \dots, v_p$ are vectors in $\mathbf{R}^n$, and w is a vector in $\mathbf{R}^n$.

Definition

We call w a **linear combination** of the vectors $v_1, v_2, \dots, v_p$. The scalars $c_1, c_2, \dots, c_p$ are called the **weights** or **coefficients**.

Example



Let $v = \begin{pmatrix}1\\2\end{pmatrix}$ and $w = \begin{pmatrix}1\\0\end{pmatrix}$.

What are some linear combinations of v and w?

- $v + w$
- $v - w$
- $2v + 0w$
- $2w$
- $-v$

[interactive: 2 vectors] [interactive: 3 vectors]

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https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,1,-1&v3=-1,1,4&range=5&captions=combo&nomove=true

You Try It!

Example. If v=[1;2] and w=[1;0] then find

- v+w $= \begin{bmatrix}2\\2\end{bmatrix}$
- v-w $= \begin{bmatrix}0\\2\end{bmatrix}$
- 2v+0w $= 2v = 2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$
- 2w $= \begin{bmatrix}2\\0\end{bmatrix}$
- -v $= \begin{bmatrix}-1\\-2\end{bmatrix}$

Poll

Poll

Is there any vector in $\mathbf{R}^2$ that is *not* a linear combination of v and w?

$$\begin{bmatrix}a\\b\end{bmatrix} = x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}1\\0\end{bmatrix}$$

This will always work (can always find x, y scalars that work) no matter how a,b is chosen.

Poll

Poll

Is there any vector in $\mathbf{R}^2$ that is *not* a linear combination of v and w?

More Examples

What are some linear combinations of $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

What are some linear combinations of $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

$$3\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 10 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 18 \\ 9 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

$$v + 2w = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 0v + 0w = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 2v + 5w = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

every possible like $\begin{bmatrix} a \\ a \end{bmatrix}$

$$x\begin{bmatrix} 2 \\ 2 \end{bmatrix} + y\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{pmatrix} 2x - y \\ 2x - y \end{pmatrix} = (2x - y)\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Section 2.2

Vector Equations and Spans

Systems of Linear Equations

Solve the following system of linear equations:

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3. \end{aligned}$$

We can write all three equations at once as vectors:

$$\begin{pmatrix} x - y \\ 2x - 2y \\ 6x - y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

We can write this as a linear combination:

$$x\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

So we are asking:

Question: Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

$$\begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix} = \begin{bmatrix} x-y \\ 2x-2y \\ 6x-y \end{bmatrix} = \begin{bmatrix} x \\ 2x \\ 6x \end{bmatrix} + \begin{bmatrix} -y \\ -2y \\ -y \end{bmatrix}$$

$$= x\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y\begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$$

Summary

The **vector equation**

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b,$$

where $v_1, v_2, \ldots, v_p, b$ are vectors in $\mathbf{R}^n$ and $x_1, x_2, \ldots, x_p$ are scalars, has the same solution set as the linear system with augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array}\right),$$

where the v_i's and b are the columns of the matrix.

This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

Definition

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\text{Span}\{v_1, v_2, \ldots, v_p\}$. In symbols:

"the set of" "such that"

$$\boxed{\text{Span}\{v_1, v_2, \ldots, v_p\} = \{x_1 v_1 + x_2 v_2 + \cdots + x_p v_p \mid x_1, x_2, \ldots, x_p \text{ in } \mathbf{R}\}.}$$

Synonyms: $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is the subset **spanned by** or **generated by** $v_1, v_2, \ldots, v_p$.

Span{v, w} is the set of all linear combinations of v & w

Now we have several equivalent ways of making the same statement:

1. A vector b is in the span of $v_1, v_2, \ldots, v_p$.
2. The vector equation
$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$
has a solution.
3. The linear system with augmented matrix
$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array}\right)$$
is consistent.

Sanity check

$$1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \stackrel{?}{=} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \checkmark$$

Example: Is the vector b in the span of v and w, where b=[1;2;1], v=[1;0;-1] and w=[0;1;1]?

Q: Is $b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ in the span of $v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ & $w = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$?

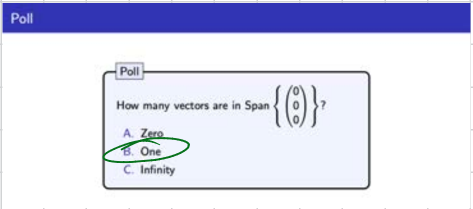
a.k.a. $x\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$\left\{\begin{aligned} x &= 1 \\ y &= 2 \\ 0 &= 0 \end{aligned}\right.$$

$$\Rightarrow \begin{bmatrix} x + 0y \\ y \\ -x+y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \left\{\begin{aligned} x &= 1 \\ y &= 2 \\ -x+y &= 1 \end{aligned}\right. \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

system of lin. eqns

You Try It!



$1\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ ✓

$0\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ ✓

$-5\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ ✓

Example: If v1=[2;1], what is Span{v1,3v1}?

Q_1: span$\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}6\\3\end{bmatrix}\right\}$ what is it?

Q_0: what are some vectors in it?

$$x\begin{bmatrix}2\\1\end{bmatrix}+y\begin{bmatrix}6\\3\end{bmatrix}=x\begin{bmatrix}2\\1\end{bmatrix}+3y\begin{bmatrix}2\\1\end{bmatrix}$$
$$=(x+3y)\begin{bmatrix}2\\1\end{bmatrix}$$

$$\boxed{A:\ \text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}6\\3\end{bmatrix}\right\}=\text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}}$$

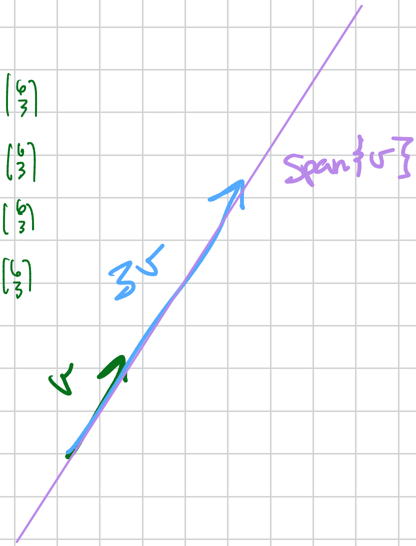
✓

$\begin{bmatrix}3\\3/2\end{bmatrix}=\frac{3}{2}\begin{bmatrix}2\\1\end{bmatrix}+0\begin{bmatrix}6\\3\end{bmatrix}$

$\begin{bmatrix}30\\15\end{bmatrix}=3\begin{bmatrix}2\\1\end{bmatrix}+4\begin{bmatrix}6\\3\end{bmatrix}$

$\begin{bmatrix}2\\1\end{bmatrix}=1\begin{bmatrix}2\\1\end{bmatrix}+0\begin{bmatrix}6\\3\end{bmatrix}$

$\begin{bmatrix}-4\\-2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}-\begin{bmatrix}6\\3\end{bmatrix}$



Example: Solve the system and write out the corresponding vector equation. How can the system be interpreted geometrically?

$$\begin{cases}x-y=8\\2x-2y=16\end{cases}$$
6x-y=3

$$\Rightarrow\begin{bmatrix}x-y\\2x-2y\end{bmatrix}=\begin{bmatrix}8\\16\end{bmatrix}\Rightarrow\begin{bmatrix}x\\2x\end{bmatrix}+\begin{bmatrix}-y\\-2y\end{bmatrix}=\begin{bmatrix}8\\16\end{bmatrix}\Rightarrow\boxed{x\begin{bmatrix}1\\2\end{bmatrix}+y\begin{bmatrix}-1\\-2\end{bmatrix}=\begin{bmatrix}8\\16\end{bmatrix}}$$

Q: consistent? yes!

Q: How many soln? infinitely many.

Example: What is Span{[1;-1;0],[2;7;0],[10;-6;0]}?

Q_1: $\text{Span}\left\{ \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}, \begin{bmatrix}10\\-6\\0\end{bmatrix} \right\}$ what is it?

Q_2: (you try it!) write down a few vectors in the span.

Need to pick scalars x, y, z then compute

$$x\begin{bmatrix}1\\-1\\0\end{bmatrix} + y\begin{bmatrix}2\\7\\0\end{bmatrix} + z\begin{bmatrix}10\\-6\\0\end{bmatrix} = \begin{bmatrix}*\\*\\*\end{bmatrix}$$

tell me these.

$$\begin{bmatrix}13\\0\\0\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}, \begin{bmatrix}65\\21\\0\end{bmatrix}, \begin{bmatrix}21\\-13\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\-2\\0\end{bmatrix}, \begin{bmatrix}3\\-3\\0\end{bmatrix}, \dots, \begin{bmatrix}a\\b\\0\end{bmatrix}$$

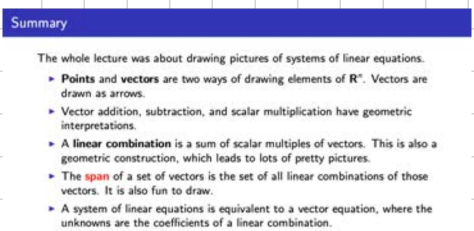
is every vector $\begin{bmatrix}a\\b\\0\end{bmatrix}$ possible?

$$x\begin{bmatrix}1\\-1\\0\end{bmatrix} + y\begin{bmatrix}2\\7\\0\end{bmatrix} + z\begin{bmatrix}10\\-6\\0\end{bmatrix} = \begin{bmatrix}a\\b\\0\end{bmatrix}$$

$\boxed{\text{consistent?}}$ for any choice of a, b.

no pivot in aug column!!

$$\Rightarrow \left[\begin{array}{ccc|c}1 & 2 & 10 & a\\ -1 & 7 & -6 & b\\ 0 & 0 & 0 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 2 & 10 & a\\ 0 & 9 & 4 & a+b\\ 0 & 0 & 0 & 0\end{array}\right]$$


Summary

The whole lecture was about drawing pictures of systems of linear equations.

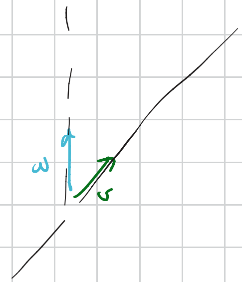
- **Points** and **vectors** are two ways of drawing elements of $\mathbf{R}^n$. Vectors are drawn as arrows.
- Vector addition, subtraction, and scalar multiplication have geometric interpretations.
- A **linear combination** is a sum of scalar multiples of vectors. This is also a geometric construction, which leads to lots of pretty pictures.
- The span of a set of vectors is the set of all linear combinations of those vectors. It is also fun to draw.
- A system of linear equations is equivalent to a vector equation, where the unknowns are the coefficients of a linear combination.

<u>yes</u>. $\begin{bmatrix}a\\b\\0\end{bmatrix}$ is always in the $\text{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}, \begin{bmatrix}10\\-6\\0\end{bmatrix}\right\}$.

What does span look like?

① $\text{Span}\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}6\\3\end{bmatrix}\right\}$ looks like a line.

Same line as $\text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$.



② $\text{Span}\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$ looks like $\mathbb{R}^2$

③ $\text{Span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}, \begin{bmatrix}10\\-6\\0\end{bmatrix}\right\}$ looks like "The floor" a plane in $\mathbb{R}^3$.