Section 2.3

Matrix Equations

Definition

The **product** of A with a vector x in $\mathbf{R}^n$ is the linear combination

$$Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \overset{\text{def}}{=} x_1v_1 + x_2v_2 + \cdots + x_nv_n.$$

this means the equality is a definition

these must be equal

The output is a vector in $\mathbf{R}^m$.

Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1\begin{bmatrix} 4 \\ 7 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 8 \end{bmatrix} + 3\begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 32 \\ 50 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2(5)+3(6) \\ 7+2(8)+3(9) \end{bmatrix} = \begin{bmatrix} 32 \\ 50 \end{bmatrix}$$

So $$\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 50 \end{bmatrix} \checkmark$$

Note that the number of columns of A has to equal the number of rows of x.

Matrix Equations
An example

Question

Let v_1, v_2, v_3 be vectors in $\mathbf{R}^3$. How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

Q: how many entries on v_1, v_2, v_3? in $\mathbb{R}^2$? in $\mathbb{R}^3$?

Matrix Equations
In general

Let $v_1, v_2, \ldots, v_n$, and b be vectors in $\mathbf{R}^m$. Consider the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_nv_n = b.$$

It is equivalent to the **matrix equation**

$$Ax = b$$

where

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

m = # rows of A b in $\mathbb{R}^m$

n = # cols of A x in $\mathbb{R}^n$

Conversely, if A is any $m \times n$ matrix, then

$Ax = b$ is equivalent to the vector equation $x_1v_1 + x_2v_2 + \cdots + x_nv_n = b$

where $v_1, \ldots, v_n$ are the columns of A, and $x_1, \ldots, x_n$ are the entries of x.

Linear Systems, Vector Equations, Matrix Equations, ...

We now have *four* equivalent ways of writing (and thinking about) linear systems:

1. As a system of equations:
$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5 \end{aligned}$$
2. As an augmented matrix:
$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right)$$
3. As a vector equation ($x_1 v_1 + \cdots + x_n v_n = b$):
$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$
4. As a matrix equation ($Ax = b$):
$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

We will move back and forth freely between these over and over again, for the rest of the semester. Get comfortable with them now!

In particular, *all four have the same solution set.*

Very Important Fact That Will Appear on Every Midterm and the Final

$Ax = b$ has a solution

$\iff$ there exist $x_1, \ldots, x_n$ such that $A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b$

"if and only if"

$\iff$ there exist $x_1, \ldots, x_n$ such that $x_1 v_1 + \cdots + x_n v_n = b$

$\iff$ b is a linear combination of $v_1, \ldots, v_n$

$\iff$ b is in the span of the columns of A.

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,0,-1&target=0,2,2&tlabel=b&range=5

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?

When Solutions Always Exist

Here are criteria for a linear system to *always* have a solution.

Theorem

Let A be an $m \times n$ (non-augmented) matrix. The following are equivalent:

1. $Ax = b$ has a solution *for all* b in $\mathbf{R}^m$.
2. The span of the columns of A is all of $\mathbf{R}^m$.
3. A has a pivot in each row.

recall that this means that for given A, either they're all true, or they're all false

Example where conditions ARE satisfied

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,0,-1&v3=-1,2,2&range=5&capopt=matrix

Example where conditions are NOT satisfied

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,0,-1&v3=1,-1,2&range=5&capopt=matrix

Properties of the Matrix–Vector Product

Let c be a scalar, u, v be vectors, and A a matrix.

- $A(u + v) = Au + Av$
- $A(cv) = cAv$

Consequence: If u and v are solutions to $Ax = 0$, then so is every vector in Span$\{u, v\}$. Why?

Important

The set of solutions to $Ax = 0$ is a span.

Summary

- We have four equivalent ways of writing a system of linear equations:
  1. As a system of equations.
  2. As an augmented matrix.
  3. As a vector equation.
  4. As a matrix equation $Ax = b$.
- $Ax = b$ is consistent if and only if b is in the span of the columns of A. The latter condition is geometric: you can draw pictures of it.
- $Ax = b$ is consistent for all b in $\mathbf{R}^m$ if and only if the columns of A span $\mathbf{R}^m$.

Section 2.4

Solution Sets

Plan For Today

Today we will learn to describe and draw the solution set of an arbitrary system of linear equations $Ax = b$, using spans.

Recall: the **solution set** is the collection of all vectors x such that $Ax = b$ is true.

Homogeneous Systems

Definition
A system of linear equations of the form $Ax = 0$ is called **homogeneous**.

Definition
A system of linear equations of the form $Ax = b$ with $b \neq 0$ is called **inhomogeneous.**

A homogeneous system always has the solution $x = 0$. This is called the **trivial solution.** The nonzero solutions are called **nontrivial.**

Observation
$Ax = 0$ has a nontrivial solution
$\iff$ there is a free variable
$\iff$ A has a column with no pivot.

Question
What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}?$$

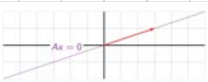
Observation
Since the last column (everything to the right of the =) was zero to begin, it will always stay zero! So it's not really necessary to write augmented matrices in the homogeneous case.

Homogeneous Systems
Example

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$



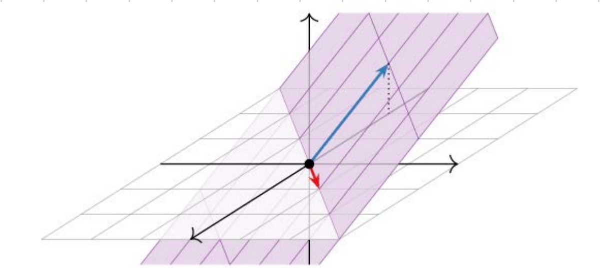
Note: *one free variable means the solution set is a line in* $\mathbf{R}^2$ ($2 = \#$ variables $= \#$ columns).

https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?lock=true&x=3,1&mat=1,-3:2,-6&range2=5

Homogeneous Systems
Example

Question
What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}?$$



https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?lock=true&x=0,0,0

Note: *two* free variables means the solution set is a *plane* in $\mathbf{R}^3$ ($3 = \#$ variables $= \#$ columns).

Question

What is the solution set of $Ax = 0$, where $A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$

Note: *two free variables means the solution set is a plane* in $\mathbf{R}^4$ ($4 = \#$ variables $= \#$ columns).

Parametric Vector Form

Homogeneous systems

Let A be an $m \times n$ matrix. Suppose that the free variables in the homogeneous equation $Ax = 0$ are, for example, x_3, x_6, and x_8.

1. Find the reduced row echelon form of A.
2. Write the parametric form of the solution set, including the redundant equations $x_3 = x_3$, $x_6 = x_6$, and $x_8 = x_8$. Put equations for all of the x_i in order.
3. Make a single vector equation from these equations by putting x_3, x_6, and x_8 as coefficients of vectors v_3, v_6, and v_8, respectively.

The solutions to $Ax = 0$ will then be expressed in the form

$$x = x_3 v_3 + x_6 v_6 + x_8 v_8$$

for some vectors v_3, v_6, v_8 in $\mathbf{R}^n$, and any scalars x_3, x_6, x_8.

In this case, the solution set to $Ax = 0$ is

$$\text{Span}\{v_3, v_6, v_8\}.$$

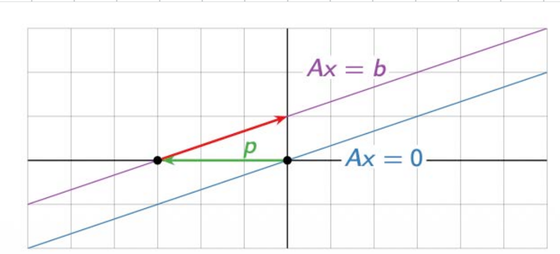
Inhomogeneous Systems
Example

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?x=-3,0&mat=1,-3:2,-6&lock=true&closed=true

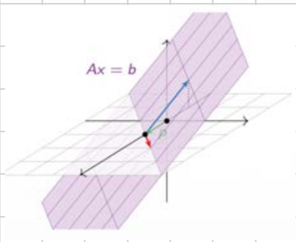


Inhomogeneous Systems
Example

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}?$$



Homogeneous vs. Inhomogeneous Systems

Key Observation

The set of solutions to $Ax = b$, if it is nonempty, is obtained by taking one **specific** or **particular solution** p to $Ax = b$, and adding all solutions to $Ax = 0$.

Very Important

Let A be an $m \times n$ matrix. There are now two *completely different* things you know how to describe using spans:

- The **solution set:** for fixed b, this is all x such that $Ax = b$.
  - This is a span if $b = 0$, or a translate of a span in general (if it's consistent).
  - Lives in $\mathbf{R}^n$.
  - Computed by finding the parametric vector form.
- The **span of the columns:** this is all b such that $Ax = b$ is consistent.
  - This is the span of the columns of A.
  - Lives in $\mathbf{R}^m$.

Summary

- The solution set to a **homogeneous** system $Ax = 0$ is a span. It always contains the **trivial solution** $x = 0$.
- The solution set to a **nonhomogeneous** system $Ax = b$ is either empty, or it is a translate of a span: namely, it is a translate of the solution set of $Ax = 0$.
- The solution set to $Ax = b$ can be expressed as a translate of a span by computing the **parametric vector form** of the solution.
- The solution set to $Ax = b$ and the span of the columns of A (from the previous lecture) are two completely different things, and you have to understand them separately.