2026/06/03

Agenda	Announcements
* 2.5	* midterm 1 in $\boxed{\text{Lecture hall}}$ Friday 2:00 - 3:15 PM

Warm up

(1) What is the parametric form of the general solution to a system of eqs whose corresponding aug matrix in RREF is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right)$$

$x + y + z = 1$

$x = 1 - y - z$
$y = y$
$z = z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-y-z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\text{Span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$

(2) (T/F)

Is the 0 vector is a solution to a matrix equation, then the matrix equation must be homogeneous

$A\vec{0} = \vec{0}$ True

Review

* The solution set to a homogeneous system $Ax=0$ is a span.
always contains the trivial solution $x=0$

* The solution set to $Ax=b$ can be expressed as a translate

of a span.

Solution set vs Span of Columns (very important)

Let A be an $m \times n$ matrix.

① The solution set. For fixed b, this is all x, $Ax=b$

* This is a span if $b=0$, or a translate of a span in general.
* lives in $\mathbb{R}^n$
* computed by finding the param. vector. form

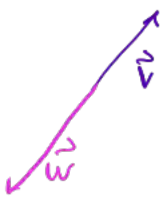
② The span of the columns: this is all b s.t. $Ax=b$ is consistent

* This is the span of cols of A
* lives in $\mathbb{R}^m$

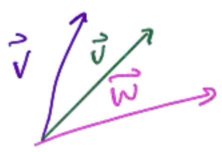
2.5

Motivation

span$\{\vec{v},\vec{w}\}$



span$\{\vec{v},\vec{u},\vec{w}\}$



One of the vectors (at least) is redundant.

Linear Independence

Definition

A set of vectors $\{v_1, \dots, v_p\}$ in $\mathbb{R}^n$ is linearly independent is the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution $(x_1 = \dots = x_p = 0)$

The set $\{v_1, \dots, v_p\}$ is linearly dependent otherwise.

In other words $\{v_1, \dots, v_p\}$ is linearly dependent if there exist numbers $x_1, \dots, x_p$, not all zero s.t.

$$x_1 v_1 + \dots + x_p v_p = 0.$$

This is called the linear dependence relation.

Q: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent

Equiv, does the (homogeneous) vector eq

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

have a nontrivial solution?

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so, $x = -2z$, $y = -z$, so there is a nontrivial solution, so vectors are linearly dependent

Q: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Equiv, does the homog. vector eq

$$x \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

have a nontrivial sol?

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -2 & 2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The trivial sol is the only sol, so vectors are linearly indep.

Linear independence & matrix columns

Recall, $\{v_1, \dots, v_p\}$ linearly indep iff

$$x_1 v_1 + \dots + x_p v_p = 0$$

only has the trivial sol $(x_1 = \dots = x_p = 0)$

Equivalently, iff

$$Ax = 0$$

only has the trivial sol

where

$$A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_p \\ | & & | \end{pmatrix}$$

This is true iff the matrix A has a pivot in each column.

Important

* The vectors $v_1, \ldots, v_p$ are lin. indep. iff the matrix with cols $v_1, \ldots, v_p$ has a pivot in each column

Criterion

Suppose that one of the vectors $\{v_1, \ldots, v_p\}$ is a linear combination of the other ones

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$

then the vectors are linearly dependent

$$2v_1 - \frac{1}{2}v_2 - v_3 + 6v_4 = 0$$

Conversely, if the vectors are linearly dependent

$$2v_1 - \frac{1}{2}v_2 + 6v_4 = 0.$$

then one vector is a linear comb (in the span of) the other ones

$$V_2 = 4V_1 + 12V_4$$

Theorem

A set of vectors $\{v_1, \dots, v_p\}$ is linearly dependent if and only if one of the vectors is in the span of the other ones

Theorem

A set of vectors $\{v_1, \dots, v_p\}$ is linearly dependent if and only if you can remove one of the vectors without shrinking the span

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Theorem

A set of vectors $\{v_1, \dots, v_p\}$, $v_1 \neq \vec{0}$

then $\{v_1, \dots, v_p\}$ is linearly dependent

if and only if there is some j such that

v_j is in the span of $\{v_1, \dots, v_{j-1}\}$

Equivalently, $\{v_1, \dots, v_p\}$ is linearly independent

if for every j, v_j is not in the span $\{v_1, \dots, v_{j-1}\}$

remark

a set of (nonzero) vectors is linearly indep iff every time you add another vector to the set, the span gets bigger.

Why: Take largest j s.t. v_j is in the span of others

Then v_j is in the span of $v_1, \dots, v_{j-1}$. If not, $(j=3)$

$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$ | $v_4 = -\frac{1}{6}(2v_1 - \frac{1}{2}v_2 - v_3)$

Q: Are there four vectors u, v, w, x in $\mathbb{R}^3$

which are linearly dependent, but such that

u is not a linear combination of v, w, x?

Yes, see picture.

Theorem

Let $v_1, \dots, v_p$ be vectors in $\mathbb{R}^n$,

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}$$

Then you can delete cols of A without pivots

without changing the span $\{v_1, \dots, v_p\}$

Remark

Let d be # of pivot columns

* If $d=1$, span $\{v_1, \dots, v_p\}$ is a line.

  $d=2$ plane

  $d=3$ 3-space

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why? | If matrix is in RREF

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

the non pivot col is in the span of the pivot cols.

$$\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

& pivot cols are linearly indep

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow x_1 = x_2 = x_4 = 0.$$

Fact 1

say $v_1, \ldots v_n$ are in $\mathbb{R}^m$. If $n > m$

then $\{v_1, \ldots, v_n\}$ is linearly dependent

$$A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \quad n > m$$

cannot have a pivot in each col

max n linearly indep vectors in $\mathbb{R}^n$

* A wide matrix can't have linearly indep columns.

Fact 2

If one of $v_1, \ldots, v_n$ is zero.

then $\{v_1, \ldots, v_n\}$ is linearly dependent

adding $\vec{0}$ does not increase span

say $v_1 = 0$

$3v_1 + 0v_2 + \cdots + 0v_n = 0$

* a set containing the zero vector is lin. dependent.

Finishing up 2.4

Q: what is the solution set of $Ax = b$

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad \& \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \; ?$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ -2 & 2 & -4 & -2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x - y + 2z = 1$$

$$x = 1 + y - 2z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 + y - 2z \\ y \\ z \end{pmatrix}$$

$$= \underbrace{y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}_{\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$